# THE RELAXATION-TIME WIGNER EQUATION

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Abstract. The relaxation–time (RT) Wigner equation models the quantum–mechanical motion of electrons in an electrostatic field, including their interaction with phonons. We discuss the conditions on a Wigner distribution function for being 'physical', and show that they will stay 'physical' under temporal evolution. Particular attention is paid to the proper definition of the particle density for Wigner functions $w \notin L^1$. For the $1D$–periodic, self–consistent RT–Wigner–Poisson equation we give a local convergence result towards the steady state.

**1. Introduction.** This paper is concerned with the analysis of the relaxation–time Wigner equation and the physical properties of its solution. The Wigner formalism, which represents a phase–space description of quantum mechanics, has in recent years attracted considerable attention of solid state physicists for including quantum effects into the simulation of ultra–integrated semiconductor devices, like resonant tunneling diodes, e.g. ([7], [10], [5]). Also, the Wigner (–Poisson) equation has recently been the objective of a detailed mathematical analysis. For a physical derivation and the discussion of many of its analytical properties we refer the reader to [15], [12], [6] (and references therein).

The real–valued Wigner (quasi) distribution function $w = w(x, v, t)$ describes the state of an electron ensemble in the $2d$–dimensional position–velocity $(x, v)$–phase space. In the absence of collision and scattering, and in the effective–mass approximation, its time evolution under the action of the (real–valued) electrostatic potential $V(x, t)$ is governed by the Wigner equation, which reads in scaled form:

$$w_t + v \cdot \nabla_x w - \Theta[V]w = 0, \quad x, v \in \mathbb{R}^d, \quad d = 1, 2 \text{ or } 3, \tag{1.1}$$

with the pseudo-differential operator

$$\begin{aligned} \Theta[V]w &= i\delta V\left(x, \frac{1}{i}\nabla_v, t\right)w = \frac{i}{(2\pi)^d}\int_{\mathbb{R}^d_\eta}\int_{\mathbb{R}^d_{v'}} \delta V(x,\eta,t)w(x,v',t)e^{i(v-v')\cdot\eta}dv'd\eta, \\ \delta V(x,\eta,t) &= V\left(x+\frac{\eta}{2},t\right) - V\left(x-\frac{\eta}{2},t\right). \end{aligned} \tag{1.2}$$

In order to account for electron-electron interactions in a simple mean-field approximation (1.1) has to be coupled to the Poisson equation

$$\Delta V(x, t) = D(x) - n(x, t), \tag{1.3}$$

where $D$ denotes the doping profile of the semiconductor. In this kinetic framework the particle density $n$ is (formally) defined as $n = \int w\, dv$.

The relaxation–time ($RT$) approximation is the simplest model to account for electron–phonon scattering, but it still yields remarkable results in device simulations. In [10] the

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relaxation term $1/\tau\ [w_0(x,v) - w(x,v,t)]$ serves as the right hand side of (1.1), coupled to (1.3). In the application to semiconductor device simulations $\tau$ is often modeled as a positive function of $x$ and $v$. However, as part of the analysis of §2 requires $\tau$ to be constant, we will sometimes restrict ourselves to this case. $w_0$ is usually chosen as a steady state of the Wigner–Poisson system, satisfying

$$v \cdot \nabla_x w_0 - \Theta[V_0] w_0 = 0, \quad \Delta V_0 = D - n_0, \tag{1.4}$$

and $n_0$ is the particle density pertaining to $w_0$. $w_0$ describes the state of the phonons in thermodynamic equilibrium. It is uniquely determined only when prescribing a thermodynamic distribution function for the included pure quantum states ([1, 16]).

The outline of this paper is as follows: in §2 we discuss the two equivalent descriptions of physical quantum states by means of Wigner functions and density matrices, and then analyze the properties of the RT–Wigner equation. In §3 we will study the one-dimensional, periodic RT–Wigner–Poisson system, and obtain a local convergence result for large time.

## 2. Physical Quantum States: Time Evolution.

In this section, we will analyze the time evolution of physically relevant initial states $w^I$ under the RT–Wigner equation for a given electrostatic potential $V(x,t)$:

$$\begin{aligned} &w_t + v \cdot \nabla_x w - \Theta[V] w = -\frac{w - w_0}{\tau}, \quad t > 0, \\ &w(x,v,t=0) = w^I(x,v), \quad x, v \in \mathbb{R}^d. \end{aligned} \tag{2.1}$$

To start with, we state an existence and uniqueness result for initial Wigner functions in $L^2(\mathbb{R}^{2d})$, the space which contains the physical quantum states. It is a simple extension of the analysis for the linear Wigner equation (Th. 1, Lemma 5 in [14]), which is based on perturbation arguments for $C_0$ semigroups (see [18]).

**Proposition 2.1.** a) *Let* $w^I$, $w_0 \in L^2(\mathbb{R}^d_x \times \mathbb{R}^d_v)$, $V \in L^\infty(\mathbb{R}^d_x \times \mathbb{R}^+_t)$, *and* $\tau(x,v) \geq \tau_0 > 0$, *then* (2.1) *has a unique mild solution* $w \in C([0,\infty); L^2(\mathbb{R}^d_x \times \mathbb{R}^d_v))$. b) *If, additionally,* $v \cdot \nabla_x w^I \in L^2(\mathbb{R}^{2d})$ *and* $V \in C^1([0,\infty); L^\infty(\mathbb{R}^d_x))$, *then* $w$ *is the unique classical solution.*

If the potential $V$ is constant in time and $w_0$ is a steady state Wigner function for the same potential, then it is seen immediately that $w(t) \to w_0$ in $L^2(\mathbb{R}^{2d})$ as $t \to \infty$.

Due to the definition of the pseudo-differential operator $\Theta[V]$ in terms of Fourier transforms, $L^2(\mathbb{R}^{2d})$ is the natural framework to analyze the (RT-) Wigner equation. In this setting, however, the following definition of the particle density, inspired by analogy with classical kinetic theory,

$$n(x,t) = \int_{\mathbb{R}^d} w(x,v,t)\,dv \tag{2.2}$$

is only formal. Also, in contrast to classical phase space distribution functions, Wigner functions need not be pointwise nonnegative. Therefore, the preservation of nonnegativity of the physically observable quantity $n$ is not obvious.

In order to analyze the time evolution of physically relevant states, we will have to exploit the equivalent descriptions of quantum states via density matrices and Wigner functions. Through this one-to-one equivalence we will also be able to give a meaning to (2.2). We are now going to define quantum states that have a well defined and nonnegative particle density $n \in L^1_+(\mathbb{R}^d_x)$. (This exposition will closely follow [13] and §2 of [12].)

A physically relevant, mixed quantum state is uniquely described by a positive, self–adjoint, trace class operator $\widehat{\rho}$ acting on $L^2(\mathbb{R}^d_x)$. In the sequel we will refer to it as 'density matrix operator', and thus tacitly assume it to have the previously listed properties. $\widehat{\rho}$ is Hilbert-Schmidt and can be represented as an integral operator on $L^2(\mathbb{R}^d_x)$:

$$(2.3)\qquad (\widehat{\rho} f)(x) = \int_{\mathbb{R}^d} \rho(x,y) f(y) dy, \quad f \in L^2(\mathbb{R}^d),$$

Its kernel $\rho \in L^2(\mathbb{R}^d_x \times \mathbb{R}^d_y)$ satisfies $\rho(x,y) = \overline{\rho(y,x)}$, $x$ and $y$ being two position variables. $\rho$ possesses a diagonal Fourier expansion (convergent in $L^2(\mathbb{R}^{2d})$) of the form

$$(2.4)\qquad \rho(x,y) = \sum_{j\in\mathbb{N}} \lambda_j \psi_j(x) \overline{\psi_j}(y),$$

where $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^2(\mathbb{N})$ and the complete o.n.s. $\{\psi_j\}_{j\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$ are the eigenvalues and eigenfunctions of $\widehat{\rho}$, respectively. Since $\widehat{\rho}$ is positive and trace class we have $\lambda_j \geq 0$, $j \in \mathbb{N}$ and $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N})$. Here $\lambda_j$ represents the occupation probability of the pure quantum state $\psi_j$ within the considered mixed state.

We will now turn to the definition of the particle density associated with $\widehat{\rho}$ or, equivalently, its kernel $\rho(x,y)$. In the framework of Schrödinger wave functions the particle density is defined as

$$(2.5)\qquad n(x) = \sum_{j\in\mathbb{N}} \lambda_j |\psi_j(x)|^2$$

which, under the above assumptions, clearly lies in $L^1_+(\mathbb{R}^d)$. Because of (2.4) the particle density is therefore formally obtained as $\rho(x,x)$. But as the subspace $\{x = y\}$ of $\mathbb{R}^{2d}$ is of measure zero, this expression is not immediately meaningful. To clarify the situation we have to recall that $\rho$ is not simply an $L^2$-function, but the kernel of a density matrix operator. We now introduce the auxiliary function

$$(2.6)\qquad z(x,\eta) = \rho\left(x + \frac{\eta}{2}, x - \frac{\eta}{2}\right), \quad z \in L^2(\mathbb{R}^{2d}),$$

which permits the following characterization of the trace class operators under consideration (see [12]).

**Lemma 2.2.**
*Let $\widehat{\rho}$ be a positive, self–adjoint, Hilbert-Schmidt operator on $L^2(\mathbb{R}^d_x)$. Then $\widehat{\rho}$ is trace class iff $z \in C_0(\mathbb{R}^d_\eta; L^1(\mathbb{R}^d_x))$, which then also implies $z \in C_0(\mathbb{R}^d_x; L^1(\mathbb{R}^d_\eta))$.*

Since this will be an essential part of the subsequent analysis we will now briefly illustrate this result and its proof:

a) We first assume that the considered $\widehat{\rho}$ has a kernel satisfying $z \in C_0(\mathbb{R}^d_\eta; L^1(\mathbb{R}^d_x)) \cap C(\mathbb{R}^{2d})$. Then result 2 on p. 114 of [8] asserts that $\widehat{\rho}$ is trace class with $Tr\widehat{\rho} = \int z(x,0)dx$. By density this then extends to $z \in C_0(\mathbb{R}^d_\eta; L^1(\mathbb{R}^d_x))$.

b) For any trace class operator $\widehat{\rho}$ on $L^2(\mathbb{R}^d_x)$ the following embedding result has been obtained in [3]: $z \in C_0(\mathbb{R}^d_\eta; L^1(\mathbb{R}^d_x))$ with the estimate

$$(2.7)\qquad \|z\|_{L^\infty(\mathbb{R}^d_\eta; L^1(\mathbb{R}^d_x))} \leq Tr|\widehat{\rho}|.$$

The particle density associated with the density matrix operator $\widehat{\rho}$ is therefore obtained as the restriction of $z$ to the subspace $\{\eta = 0\}$, or equivalently by appropriately smoothing $\rho$ across its diagonal $x = y$:

$$(2.8)\qquad n(x) = z(x,0) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} \rho\left(x + \frac{\eta}{2}, x - \frac{\eta}{2}\right) \frac{e^{-|\eta|^2/2\varepsilon}}{(2\pi\varepsilon)^{d/2}} d\eta \in L^1_+(\mathbb{R}^d_x).$$

Clearly, $\rho(x,y)$ could also be smoothed by other sequences of decay function converging to the $\delta$-distribution (see [3]). The trace of the density matrix operator satisfies

$$(2.9)\qquad Tr\widehat{\rho} = \sum_{j\in\mathbb{N}} \lambda_j = \int_{\mathbb{R}^d} n(x) dx.$$

In the sequel we will completely identify a density matrix operator $\widehat{\rho}$ and its kernel $\rho$ and simply call them 'density matrix'.

We will next introduce the Wigner transform of a density matrix and discuss the properties it inherits. The Wigner function corresponding to a density matrix $\widehat{\rho}$, in the sequel referred to as a 'physical Wigner function', is defined by

$$w(x,v) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} z(x,\eta) e^{-iv\cdot\eta} d\eta, \quad x,v \in \mathbb{R}^d. \tag{2.10}$$

Since $\widehat{\rho}$ is self–adjoint and Hilbert-Schmidt, the Wigner function is real valued and $w \in L^2(\mathbb{R}^d_x \times \mathbb{R}^d_v)$, with $\|w\|_2 = (4\pi)^{-d/2}\|\rho\|_2$. The positivity and the finite trace of $\widehat{\rho}$ can also be reformulated in terms of $w$, though in a somewhat more difficult fashion. As we will not need them here, we just refer to [12]. From $z \in C_0(\mathbb{R}^d_x; L^1(\mathbb{R}^d_\eta))$ one immediately deduces

$$w \in C_0(\mathbb{R}^d_x; \mathcal{F}L^1(\mathbb{R}^d_v)). \tag{2.11}$$

Also, since the quantum mechanical phase space is symmetric with respect to the $x$ and $v$ variables, one can obtain $w \in C_0(\mathbb{R}^d_v; \mathcal{F}L^1(\mathbb{R}^d_x))$. This regularity of a physical Wigner function $w$ is in general still not sufficient to justify a definition of the particle density as $n = \int w\, dv$. However, (2.8) and (2.10) show that the particle density of a physical Wigner function can be calculated through the following regularization, which converges in $L^1(\mathbb{R}^d_x)$:

$$n(x) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} w(x,v) e^{-\varepsilon|v|^2/2} dv \in L^1_+(\mathbb{R}^d). \tag{2.12}$$

For Wigner functions $w \in L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v)$ this of course coincides with $\int w\, dv$. However, $w(t) \in L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v)$ can typically not be expected when considering the time evolution (either by (1.1) or (2.1)) with a non–smooth potential $V$. The associated difficulty to define $n$ was also numerically experienced in [17], where a deterministic particle method for the Wigner–Poisson system was studied.

We will now study the time evolution of quantum states under the RT–Wigner equation, and we will show that initially 'physical Wigner function' will stay 'physical', i.e., correspond to a positive density matrix operator with finite trace. For the rest of this section we will only consider constant relaxation times $\tau > 0$ and time–independent potentials $V(x)$. Also, we will assume that the Hamiltonian $H = -\frac{1}{2}\Delta + V : D(H) \to L^2(\mathbb{R}^d)$ be essentially self–adjoint on $C_0^\infty(\mathbb{R}^d)$ (for sufficient conditions on $V$ see [19], [12], e.g.). Here we only mention a condition such that the potential is a relatively bounded perturbation of $-\frac{1}{2}\Delta$:

$$V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d), \quad p > \frac{d}{2} \text{ and } p \geq 2, \tag{2.13}$$

and then $D(H) = H^2(\mathbb{R}^d)$.

¿From the transformations (2.6), (2.10) one easily sees that the density matrix $\rho(x,y)$, corresponding to the Wigner function solution of (2.1), satisfies the RT–Heisenberg equation:

$$\begin{aligned} &i\rho_t = (H_x - H_y)\rho - \frac{i}{\tau}(\rho - \rho_0), \quad t > 0, \\ &\rho(x,y,t=0) = \rho^I(x,y), \qquad\qquad x,y \in \mathbb{R}^d. \end{aligned} \tag{2.14}$$

Here, $\rho_0$ denotes the steady state density matrix corresponding to $w_0$, and the subscripts $x$ and $y$ indicate that this operator acts, respectively, only on the $x$ and $y$ variable. The analysis of [13] and Proposition 2.1 immediately show that, for $\rho^I, \rho_0 \in L^2(\mathbb{R}^d_x \times \mathbb{R}^d_y)$, (2.14) has a unique mild solution $\rho \in C([0,\infty); L^2(\mathbb{R}^d_x \times \mathbb{R}^d_y))$. From (2.14) we easily conclude $\rho(x,y,t) = \overline{\rho(y,x,t)}$, $t \geq 0$. Hence, the density matrix will, under temporal evolution, continue to be self–adjoint and Hilbert-Schmidt.

The time evolution of the Wigner equation (without relaxation term) is equivalent to countably many Schrödinger equations for each pure quantum state ([13]). Thus, a Wigner function solution to (1.1) can be expanded in a series of pure state Wigner functions with constant-in-time occupation probabilities $\lambda_j$:

$$w(t) = \sum_{j \in \mathbb{N}} \lambda_j w_j(t) \tag{2.15}$$

Each $w_j$ corresponds via

$$w_j(x,v,t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi_j\left(x + \frac{\eta}{2}, t\right) \overline{\psi}_j\left(x - \frac{\eta}{2}, t\right) e^{-iv\cdot\eta} d\eta, \tag{2.16}$$

to a pure state wave function $\psi_j$, whose time evolution is governed by the Schrödinger equation $i\psi_t = H\psi$. This equivalence was the basis of the existence and uniqueness analysis of the Wigner–Poisson problem ([4], [9]). For the RT–Wigner equation, however, this kind of equivalence does not hold, even for modified Hamiltonians. Therefore, the analysis of the whole space RT–Wigner–Poisson equation requires a different approach, and will be the subject of a subsequent paper. We remark that a different dissipative Wigner equation model has been discussed in [11]. Since that model is based on a dissipative Hamiltonian, equivalence to a system of Schrödinger equations has been preserved.

For the subsequent analysis we now introduce the following notation: $S(t) = \exp(-i\overline{H}t)$ is the unitary $C_0$ group on $L^2(\mathbb{R}^d)$ generated by the closure of $-iH$, and $T(t) = \exp[(\overline{-iH_x + iH_y})t]$ is the unitary $C_0$ group on $L^2(\mathbb{R}^d_x \times \mathbb{R}^d_y)$ generated by the closure of $-iH_x + iH_y$. According to Th. 2.1 of [13] they are related by $T(t) = S_x(t) \otimes S_y(-t)$, $t \in \mathbb{R}$, where $\otimes$ denotes the continuous extension of a tensor product of operators, acting on $L^2(\mathbb{R}^d_x)$ and $L^2(\mathbb{R}^d_y)$, respectively. For a function $\rho \in L^2(\mathbb{R}^d_x \times \mathbb{R}^d_y)$ with diagonal representation (2.4) this then reads explicitly

$$T(t)\rho = \sum_{j \in \mathbb{N}} \lambda_j (S_x(t)\psi_j(x))(S_y(-t)\overline{\psi}_j(y)). \tag{2.17}$$

We can now formulate the main result of this section:

**Theorem 2.3.**
*Let $w^I$, $w_0$ be physical Wigner functions. Then the mild solution $w(t)$ of* (2.1) *will be a physical Wigner function for $t > 0$.*

**Proof.** Since $w \in C([0,\infty); L^2(\mathbb{R}^d_x \times \mathbb{R}^d_v))$ is real valued, we only have to prove that $w(t)$, $t > 0$ corresponds to a positive density matrix operator of trace class.

We denote the diagonal Fourier expansions of the initial and steady state density matrices by

$$\rho^I(x,y) = \sum_{j \in \mathbb{N}} \lambda_j \psi_j(x)\overline{\psi_j}(y), \quad \rho_0(x,y) = \sum_{j \in \mathbb{N}} \mu_j \sigma_j(x)\overline{\sigma_j}(y). \tag{2.18}$$

By the assumptions on $w^I$, $w_0$ we have $\{\lambda_j\}$, $\{\mu_j\} \in \ell^1_+(\mathbb{N})$, and $\{\psi_j\}$, $\{\sigma_j\}$ are complete o.n.s. in $L^2(\mathbb{R}^d)$. The density matrix corresponding to $w$ satisfies (2.14), and it can be explicitly represented by the variation-of-constants formula for inhomogeneous evolution equations (see [18]):

$$\begin{aligned} \rho(x,y,t) &= e^{-\frac{t}{\tau}} \sum_{j \in \mathbb{N}} \lambda_j (S_x(t)\psi_j(x))(\overline{S_y(t)\psi_j(y)}) \\ &+ \frac{1}{\tau}\int_0^t e^{-\frac{t-t_1}{\tau}} \sum_{j \in \mathbb{N}} \mu_j (S_x(t-t_1)\sigma_j(x))(\overline{S_y(t-t_1)\sigma_j(y)}) dt_1. \end{aligned} \tag{2.19}$$

In this form we can easily check that $\widehat{\rho}(t)$ is a positive operator. For any $f \in L^2(\mathbb{R}^d)$ we obtain

$$
\iint \rho(x,y,t)\overline{f}(x)f(y)dx\,dy = e^{-\frac{t}{\tau}} \sum_{j\in\mathbb{N}} \lambda_j \left| \int \overline{f}(x) S_x(t)\psi_j(x)dx \right|^2 + \frac{1}{\tau}\int_0^t e^{-\frac{t-t_1}{\tau}} \sum_{j\in\mathbb{N}} \mu_j \left| \int \overline{f}(x) S_x(t-t_1)\sigma_j(x)dx \right|^2 dt_1 \geq 0. \tag{2.20}
$$

As both, $\{S(t)\psi_j\}$, $\{S(t)\sigma_j\}$ also form a complete o.n.s. in $L^2(\mathbb{R}^d)$ for any $t \in \mathbb{R}$, we readily obtain the global estimate

$$
\|\rho(t)\|_2 \leq e^{-t/\tau}\|\rho^I\|_2 + (1-e^{-t/\tau})\|\rho_0\|_2 \tag{2.21}
$$

from (2.19).

Since we already know that $\widehat{\rho}(t)$ is positive we can calculate its trace in the o.n. basis $\{S(t)\psi_k\}$, e.g. Again using the representation (2.19), a short calculation gives

$$
\sum_{k\in\mathbb{N}} \iint \rho(x,y,t)(\overline{S_x(t)\psi_k(x)})(S_y(t)\psi_k(y))dx\,dy = e^{-\frac{t}{\tau}} \sum_{j\in\mathbb{N}} \lambda_j + (1-e^{-\frac{t}{\tau}}) \sum_{j\in\mathbb{N}} \mu_j, \tag{2.22}
$$

and thus

$$
Tr\widehat{\rho}(t) = e^{-t/\tau} Tr\widehat{\rho}^I + (1-e^{-t/\tau})Tr\widehat{\rho}_0. \tag{2.23}
$$

This result can easily be extended to time–dependent potentials $V$ whenever they give rise to a unitary propagator for the corresponding Hamiltonian (see, e.g. Th. X.71 of [19]). The explicit representation of $\rho$ clearly shows that the occupation probabilities of pure quantum states are time–dependent in the dynamics of the RT–Wigner model.

## 3. RT–Wigner–Poisson equation (1D periodic).

In this section we shall analyze the coupled RT–Wigner–Poisson problem and give a local convergence result to the steady state $w_0$ for small relaxation times $\tau(x,v)$. We will now study the solution of the following one–dimensional model with periodic boundary conditions:

$$
w_t + vw_x - \Theta[V]w = -\frac{w-w_0}{\tau}, \quad x\in(0,2\pi), \quad v\in\mathbb{R}, \quad t>0, \tag{3.1a}
$$

$$
w(0,v,t) = w(2\pi,v,t), \quad v\in\mathbb{R}, \quad t>0, \tag{3.1b}
$$

$$
w(t=0) = w^I, \tag{3.1c}
$$

$$
n(x,t) = \int_{\mathbb{R}} w(x,v,t)dv, \tag{3.1d}
$$

$$
V_{xx} = D(x) - n(x,t), \quad x\in(0,2\pi), \quad t>0, \tag{3.1e}
$$

$$
V(0,t) = V(2\pi,t) = 0, \quad t>0, \tag{3.1f}
$$

and the potential is extended $2\pi$–periodically outside of the interval $(0,2\pi)$, which is needed to define $\Theta[V]$. Since the used techniques will be very similar to the analysis of the periodic Wigner–Poisson equation given in §4 of [2], we will only sketch the proofs here.

In order to avoid the difficulties associated with the definition of the density $n$, as described in §2, problem (3.1) will be posed in the Hilbert space $X := L^2((0,2\pi)\times\mathbb{R}, 1+v^2)$, endowed with the norm $\|u\|_X = \|u\|_2 + \|vu\|_2$. The continuous embedding $X \hookrightarrow L^1((0,2\pi)\times\mathbb{R})$ then justifies the definition (3.1d).

We define the operator $A : D(A) \to X$ by $Au = -vu_x$ and the periodic boundary conditions are incorporated in its domain $D(A) = \{u \in X | vu_x \in X; u(0,v) = u(2\pi,v), v \in \mathbb{R}\}$. $A$ generates a $C_0$ group of isometries $\{S(t), t\in\mathbb{R}\}$ on $X$, given by $S(t)u(x,v) = \tilde{u}(x-vt,v)$, where $\tilde{u}$ denotes the $2\pi$–periodic extension of $u$. The nonlinear operator $B : X \to X$ is defined by $Bu = \Theta[V[u]]u$, where the potential $V[u]$ is the solution of the Poisson equation (3.1e), (3.1f) with periodic extension outside of the interval $(0,2\pi)$.

For the analysis of (3.1) we will need the following result from [2]. Here and in the sequel $C$ denotes generic, but not necessarily equal constants.

**Lemma 3.1.** *Let $D \in L^1(0,2\pi)$, then $B$ is of class $C^\infty$ in $X$, and it satisfies*

$$\|Bu_1 - Bu_2\|_X \le C(\|u_1\|_X + \|u_2\|_X + \|D\|_{L^1(0,2\pi)})\|u_1 - u_2\|_X. \tag{3.2}$$

Therefore, we can consider $Bu + \frac{w_0 - u}{\tau}$ as a locally Lipschitz perturbation of the generator $A$, which is the basis of the following global existence result.

**Theorem 3.2.** *Let $D \in L^1(0,2\pi)$, $w_0 \in X$, and $\tau(x,v) \ge \tau_0 > 0$.*

a) *For every $w^I \in X$, (3.1) admits a unique mild solution $w \in C([0,\infty), X)$.*
b) *If $w^I \in D(A)$, then $w$ is a classical solution with $w \in C^1([0,\infty), X)$, $w(t) \in D(A)$ for $t \ge 0$.*

**Proof.** The local–in–time result is obtained as an application of §6 in [18].

To obtain a global–in–time solution we will derive an *a–priori* estimate for $\|w(t)\|_X$, which requires the regularity of classical solutions. This estimate then carries over to the mild solution by a simple density argument. Since the analysis of §2 does not immediately carry over to the space periodic case, an $L^2$–estimate of type (2.21) has to be established again. Multiplying (3.1a) by $w$ and integrating gives (since $\Theta$ is skew symmetric):

$$\frac{1}{2}\frac{d}{dt}\|w\|_2^2 = \int_0^{2\pi}\int_{\mathbb{R}} w\frac{w_0 - w}{\tau}dv\,dx \le \frac{1}{\tau_0}\|w\|_2\|w_0\|_2, \tag{3.3}$$

and thus

$$\|w(t)\|_2 \le \|w^I\|_2 + \|w_0\|_2\frac{t}{\tau_0}. \tag{3.4}$$

A uniform bound for $\|w(t)\|_2$, analogous to (2.21), can be obtained if $\tau(x,v)$ is bounded.

For the *a–priori* estimate on $\|vw(t)\|_2$ we now consider the evolution equation for $u = vw$:

$$u_t - Au - \Theta[V(t)]u = -\Omega[V_x(t)]w(t) + \frac{vw_0 - u}{\tau}, \tag{3.5}$$

with the pseudo–differential operator

$$\Omega[V(t)] = \frac{1}{2}\left[V\left(x + \frac{1}{2i}\partial_v, t\right) + V\left(x - \frac{1}{2i}\partial_v, t\right)\right]. \tag{3.6}$$

$\Omega[V(t)]$ is a bounded operator in $L^2((0,2\pi)\times\mathbb{R})$ with $\|\Omega[V(t)]\|_2 \le \|V(t)\|_{L^\infty(0,2\pi)}$. Multiplying (3.5) by $u$ and integrating gives

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 = \int_0^{2\pi}\int_{\mathbb{R}} u\Big\{-\Omega[V_x(t)]w(t) + \frac{vw_0 - u}{\tau}\Big\}dv\,dx, \tag{3.7}$$

and hence

$$\frac{d}{dt}\|w(t)\|_X \le \|V_x(t)\|_{L^\infty(0,2\pi)}\|w(t)\|_2 + \frac{1}{\tau_0}\|w_0\|_X. \tag{3.8}$$

We now use (3.4) and the estimate

$$\|V_x[u]\|_{L^\infty(0,2\pi)} \le C(\|u\|_X + \|D\|_{L^1(0,2\pi)}), \tag{3.9}$$

which follows from the Poisson equation with periodic boundary conditions. Applying Gronwall's inequality then shows that the solution $w$ exists globally in time.

Using estimation techniques similar to those for the above existence and uniqueness result we shall now give a local convergence result towards the steady state $w_0$. We remark that steady states of the Wigner–Poisson system are not unique, and $w_0$ has to be chosen as part of the RT–model.

**Theorem 3.3.** *Let $D \in L^1(0,2\pi)$, $w^I$, $w_0 \in X$, and $\tau_1 \ge \tau(x,v) \ge \tau_0 > 0$, where $w_0$ is a steady state of (3.1), i.e.,*

$$Aw_0 + Bw_0 = 0. \tag{3.10}$$

*Then* $w(t) \xrightarrow{t\to\infty} w_0$ *in* $X$ *for* $\|w^I - w_0\|_X$ *and* $\tau_1$ *sufficiently small.*

**Proof.** Like in the proof of Theorem 3.2 a) we will first establish the convergence for classical solutions, and $w_0 \in D(A)$. The result then extends to mild solutions by a density argument.

We now consider the time evolution of $y = w - w_0$. Due to (3.10), it satisfies:

$$y_t = Ay + (Bw - Bw_0) - \frac{y}{\tau}. \tag{3.10}$$

Proceeding as before gives the estimate

$$\frac{d}{dt}\|y(t)\|_2 \le \|Bw(t) - Bw_0\|_2 - \frac{1}{\tau_1}\|y(t)\|_2. \tag{3.11}$$

With the analogous estimate for $vy(t)$ we finally get by using (3.2)

$$\begin{aligned} \frac{d}{dt}\|y(t)\|_X &\le \|Bw(t) - Bw_0\|_X - \frac{1}{\tau_1}\|y(t)\|_X \\ &\le \left[C(\|y(t)\|_X + 2\|w_0\|_X + \|D\|_{L^1(0,2\pi)}) - \frac{1}{\tau_1}\right]\|y(t)\|_X. \end{aligned} \tag{3.12}$$

The asserted convergence now follows if

$$C(\|w^I - w_0\|_X + 2\|w_0\|_X + \|D\|_{L^1(0,2\pi)}) < \frac{1}{\tau_1}, \tag{3.13}$$

with $C$ independent of $w^I$, $w_0$.

The convergence behavior of the RT–Wigner–Poisson system for large $\tau$ is currently under investigation and will be discussed in a subsequent paper.

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