# A generalized Bakry-Emery condition for non-symmetric diffusions

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**Abstract**

In this paper we present a new perturbation result for proving logarithmic Sobolev inequalities. By using a non-symmetric perturbation in linear Fokker-Planck equations the entropy method yields improved estimates for the logarithmic Sobolev constant in several cases.

## 1 Introduction

In this paper we consider the large-time behavior of the Cauchy problem for linear Fokker-Planck type equations (advection-diffusion equations) for densities $\rho(x,t)$ with $x \in \mathbb{R}^n$, $t > 0$:

$$\rho_t = L\rho := \mathrm{div}(\nabla\rho + \rho(\nabla A + \vec{F})) = \mathrm{div}\left(e^{-A}(\nabla + \vec{F})e^{A}\rho\right), \quad \rho(t=0) = \rho_I \in L^1_+(\mathbb{R}^n). \tag{1.1}$$

Due to this divergence form, $\int_{\mathbb{R}^n} \rho(x,t)dx = \int_{\mathbb{R}^n} \rho_I(x)$ for all $t > 0$. We shall assume henceforth that the intitial data is normalized so that $\int_{\mathbb{R}^n} \rho_I(x)dx = 1$. Now suppose that the vector field $\vec{F}$ and the scalar $A$ satisfy

$$\mathrm{div}(e^{-A}\vec{F}) = 0, \qquad \text{and} \qquad \int_{\mathbb{R}^n} e^{-A(x)}dx = 1. \tag{1.2}$$

Then the unique normalized steady state of (1.1) is $\rho_\infty = e^{-A} \in L^1(\mathbb{R}^n)$. Because of (1.2), $\mathrm{div}(\rho\vec{F})$ is the skew-symmetric part of the operator $L$ in $L^2(\mathbb{R}^n; \rho_\infty^{-1}dx)$ acting on $\rho$, and this skew-symmetric part anihilates the steady state $\rho_\infty$.

Here we shall assume that the data $A$, $\vec{F}$ and $\rho_I$ are such that (1.1) has a unique solution $\rho \in C([0,\infty), L^1_+(\mathbb{R}^n))$. We are interested in the possible exponential decay rate of $\rho(t)$ towards $\rho_\infty$ in relative entropy which is closely related to the hypercontractivity of the semigroup generated by $L$ and to the validity of a logarithmic Sobolev inequality w.r.t. the steady state measure $\rho_\infty$ (cf. [5, 6, 1]). This inequality would read, if it holds,

$$\int_{\mathbb{R}^n} f^2 \ln f^2 \rho_\infty dx - \left(\int_{\mathbb{R}^n} f^2\rho_\infty dx\right) \ln\left(\int_{\mathbb{R}^n} f^2\rho_\infty dx\right) \le C\int_{\mathbb{R}^n} |\nabla f|^2\rho_\infty dx \tag{1.3}$$

for some fixed $C < \infty$ all $f \in L^2(\mathbb{R}^n, \rho_\infty dx)$. Notice that $A$ enters the inequality through $\rho_\infty$, but that $\vec{F}$ does not. The question to be addressed here is whether it is ever advantageous to consider a non-reversible evolution (i.e., one with $\vec{F} \ne 0$) when attempting to establish the validity of (1.3) through the method of [2, 3, 1]. Perhaps surprisingly, the answer is yes.

Following the entropy method of [2, 3, 1] we define the *relative entropy* of $\rho(t)$ w.r.t. $\rho_\infty$ as

$$e(t) = e(\rho(t)|\rho_\infty) := \int_{\mathbb{R}^n} \ln\frac{\rho(t)}{\rho_\infty}\rho(t)dx \ge 0.$$

From (1.1) we calculate its time derivative $I(t) := e'(t)$ as

$$I(t) = I(\rho(t)|\rho_\infty) = -\int_{\mathbb{R}^n} |u|^2\rho(t)dx = -4\int_{\mathbb{R}^n} |v|^2\rho_\infty dx \le 0,$$

where $u = \nabla \ln(\rho(t)/\rho_\infty)$, and $v = \nabla\sqrt{\rho(t)/\rho_\infty}$.

Note that the *entropy production* $I(t)$ is independent of $\vec{F}$. Bakry and Emery considered the natural case in which $\vec{F} = 0$, and went on to compute the second derivative of $e(t)$, i.e., $I'(t)$. They found that when $A(x)$ is uniformly convex, i.e. it satisfies a *Bakry-Emery condition* (BEC):

(A1)
$$\exists\, \lambda_1 > 0 \text{ such that } \quad \frac{\partial^2 A}{\partial x^2} = \Big(\frac{\partial^2 A(x)}{\partial x_i \partial x_j}\Big)_{i,j=1,\dots,n} \geq \lambda_1 \mathbf{I} \quad \forall x \in \mathbb{R}^n \ ,$$

then the entropy production satisfies

$$I'(t) \geq -2\lambda_1 I(t) \tag{1.4}$$

(cf. [2, 3, 1]).

Integrating (1.4) from $t$ to $\infty$ yields the logarithmic Sobolev inequality in the form

$$e(\rho|\rho_\infty) \leq \frac{1}{2\lambda_1}|I(\rho|\rho_\infty)| \quad \forall \rho \in L^1_+(\mathbb{R}^n) \text{ with } \int_{\mathbb{R}^n} \rho\, dx = 1 \tag{1.5}$$

which is equivalent to (1.3). From (1.5) it follows that the relative entropy decays exponentially in time: $e(t) \leq e^{-2\lambda_1 t} e(0)$.

In some cases, this argument not only leads to the sharp constants in the corresponding log Sobolev inequalities; it may be used to determine the cases of equality [4]. However, in other cases, the BEC does not even yield any useful bound on the log Sobolev constant. The potential $A(x) = |x|^4$, e.g., gives rise to a hypercontractive semigroup, but it violates the BEC. Nevertheless, a log Sobolev inequality can be proven by using the Holley-Stroock perturbation lemma [7, 1], in which the invariant density $\rho_\infty$ is perturbed.

Here we shall present a different kind of perturbation result, in which $\vec{F}$ is considered as a perturbation of the symmetric part of the generator $L$. As we shall show, there are densities $\rho_\infty = e^{-A}$ such that (BEC) does not hold, but are such that (1.4) does hold for an appropriate non-zero choice of $\vec{F}$ satisfying (1.2). We remark that the sharp estimates for the entropy decay rate of non-symmetric diffusions (1.1) and their symmetric counterparts are identical, since $e(\rho|\rho_\infty)$ and $I(\rho|\rho_\infty)$ coincide in both cases. Despite this, our examples show that when employing the above entropy method, better constants in the log Sobolev inequality may be produced if one considers non-symmetric diffusions leaving the given density $\rho_\infty$ invariant.

## 2 Generalized Bakry-Emery condition

**Theorem 2.1.** *Assume that there exists a $\lambda_2 > 0$ such that*

*(A2)*
$$\frac{\partial^2 A}{\partial x^2} - \frac{\nabla \otimes \vec{F} + (\nabla \otimes \vec{F})^\top}{2} \geq \lambda_2 \mathbf{I} \quad \forall x \in \mathbb{R}^n.$$

*Then the estimate (1.4) and the LSI (1.5) (with $\lambda_1$ replaced by $\lambda_2$) hold for all normalized $\rho \in L^1_+(\mathbb{R}^n)$.*

*Proof.*

$$\begin{aligned} I' \quad &= \quad \int_{\mathbb{R}^n} \operatorname{div}(\rho_\infty \nabla \frac{\rho}{\rho_\infty})|u|^2\, dx - 2\int_{\mathbb{R}^n} u \cdot \nabla\Big(\frac{1}{\rho_\infty}\operatorname{div}(\rho_\infty \nabla \frac{\rho}{\rho_\infty})\Big)\rho_\infty\, dx \\ &- \quad \int_{\mathbb{R}^n} u^\top (\nabla \otimes \vec{F} + (\nabla \otimes \vec{F})^\top) u \rho\, dx. \end{aligned} \tag{2.1}$$

Estimating the first two terms of the r.h.s. like in §2.3 of [1] yields:

$$I' \geq 2\int_{\mathbb{R}^n} u^\top \Big(\frac{\partial^2 A}{\partial x^2} - \frac{\nabla \otimes \vec{F} + (\nabla \otimes \vec{F})^\top}{2}\Big) u \rho\, dx \geq -2\lambda_2 I. \qquad \square$$

We shall now give two examples to illustrate how the non-symmetric perturbation $\mathrm{div}(\rho\vec{F})$ can help to "improve" the constant in the LSI (1.5).

**Example 2.2.** *Let $A(x,y) = x^2 + \alpha y^2$, $\alpha > 0$ in $\mathbb{R}^2$. Then the constant $\lambda_1 = \min(2, 2\alpha)$ from (A1) is known to be the sharp log Sobolev constant (cf. [4, 1]). Hence, no perturbations can improve it: consider, e.g.*

$$\vec{F} = \mu \begin{pmatrix} \alpha y \\ -x \end{pmatrix}, \quad \mu \in \mathbb{R}.$$

*Then (A2) holds with $\lambda_2 = \alpha + 1 - |\alpha - 1|\sqrt{1 + \mu^2/4} \le \lambda_1$.*

Next we consider a case where (A2) indeed provides a better constant than (A1).

**Example 2.3.** *Let $A(x,y) = |x|^{b(x^2+y^2)} + y^2$ in $\mathbb{R}^2$ with*

$$b(s) = \begin{cases} 2 + \varepsilon_1, & 0 \le s \le \varepsilon_2 \\ 2, & s \ge 2\varepsilon_2, \end{cases} \tag{2.2}$$

*and $b \in C^2[0,\infty)$ and monotonous. Now, $\lambda_1 = 0$ since $A$ is not strictly convex in a small neighborhood of $(0,0)$. And there an appropriate $\vec{F}$ can "help": Let*

$$\vec{F} = \mu e^{A(x,y)} a((x+1)^2 + (y+1)^2) \begin{pmatrix} -y-1 \\ x+1 \end{pmatrix},$$

*with*

$$a(s) = \begin{cases} -1, & 0 \le s \le 1 \\ 0, & s \ge 4, \end{cases} \tag{2.3}$$

*and $a \in C^1[0,\infty)$ and monotonous. For $\varepsilon_1$, $\varepsilon_2$, $\mu > 0$ small enough, (A2) holds for some $\lambda_2 > 0$.*

We remark that, in our framework, the condition (A2) can only be satisfied for $n \ge 2$.

A more detailed version will appear elsewhere.

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