

On the periodic Wigner-Poisson-Fokker-Planck system

Anton Arnold¹, Jose A. Carrillo², Elidon Dhamo¹

arnold@num.uni-sb.de, carrillo@goliat.ugr.es, dhamo@num.uni-sb.de

¹ Fakultät für Mathematik und Informatik, Universität des Saarlandes, Postfach 15 11 50, D-66041 Saarbrücken, Germany

² Departamento de Matemática Aplicada, Universidad de Granada, E-18071 Granada, Spain

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Abstract

This paper is concerned with the existence and uniqueness analysis of global classical solutions of a diffusive quantum evolution equation with non-linear coupling to the Poisson equation. The main technical difficulty in the existence proof is to show that the quantum Fokker-Planck term is a semigroup-generator in a weighted L^2 -space. The potential term is then a Lipschitz-perturbation of it.

1 INTRODUCTION

The object of this paper is the analysis of the coupled Wigner-Poisson-Fokker-Planck (WFPF) system in one dimension with periodic boundary conditions. We focus on the existence and uniqueness of global-in-time solutions to this system.

Wigner functions provide a kinetic description of quantum mechanics (cf. [19]) and have recently become a valuable modeling and simulation tool in fields like semiconductor device modeling (cf. [12] and references therein), quantum Brownian motion, and quantum optics ([6, 8]). The real-valued Wigner function $w(x, v, t)$ is a probabilistic quasi-distribution function in the position-velocity (x, v) phase space for the considered quantum system at time t .

Its temporal evolution is governed by the Wigner-Fokker-Planck (WFP) equation:

$$w_t + vw_x + \Theta[V]w = \beta(vw)_v + \sigma w_{vv} + 2\gamma w_{xv} + \alpha w_{xx}, \quad t > 0, \quad (1.1)$$

on the phase space slab $x \in (0, 2\pi)$, $v \in \mathbb{R}$ with periodic boundary conditions in x :

$$w(0, v, t) = w(2\pi, v, t),$$

and the initial condition

$$w(x, v, t = 0) = w^I(x, v).$$

With a vanishing right hand side Equation (1.1) would be the (diffusion-free) Wigner equation. It describes the reversible evolution of a quantum system under the action of a (possibly time-dependent) electrostatic potential $V = V(x, t)$. Its effect enters in the equation via the pseudo-differential operator $\Theta[V]$:

$$\begin{aligned} (\Theta[V]w)(x, v, t) &= i[V(x + \frac{1}{2i}\nabla_v, t) - V(x - \frac{1}{2i}\nabla_v, t)]w(x, v, t) \\ &= \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \delta V(x, \eta, t) \mathcal{F}_v w(x, \eta, t) e^{i v \eta} d\eta \\ &= \frac{i}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \delta V(x, \eta, t) w(x, v', t) e^{i(v-v')\eta} dv' d\eta, \end{aligned} \quad (1.2)$$

where $\delta V(x, \eta, t) = V(x + \frac{\eta}{2}, t) - V(x - \frac{\eta}{2}, t)$ and $\mathcal{F}_v w$ denotes the Fourier transform of w with respect to v :

$$\mathcal{F}_v w(x, \eta, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} w(x, v', t) e^{-i v' \eta} dv'.$$

For simplicity of the notation we have here set the Planck constant, particle mass and charge equal to unity.

The right hand side of (1.1) is a Fokker-Planck type model for the non-reversible interaction of this quantum system with an environment, e.g. the interaction of electrons with a phonon bath (cf. [9, 18]). In (1.1) $\beta \geq 0$ is the friction parameter and the parameters $\alpha, \gamma \geq 0, \sigma > 0$ constitute the phase-space diffusion matrix of the system. In the kinetic Fokker-Planck equation of classical mechanics (cf. [17, 7]) one would have $\alpha = \gamma = 0$. For the WFP equation (1.1) we have to assume

$$\begin{pmatrix} \alpha & \gamma + \frac{i}{4}\beta \\ \gamma - \frac{i}{4}\beta & \sigma \end{pmatrix} \geq 0,$$

which guarantees that the system is *quantum mechanically correct*. More precisely, it guarantees that the corresponding von Neumann equation is in Lindblad form and that the density matrix of the quantum system stays a positive operator under temporal evolution (see [4] for details).

In the sequel we shall hence assume

$$\alpha\sigma \geq \gamma^2 + \frac{\beta^2}{16}. \quad (1.3)$$

However, the subsequent mathematical analysis will even hold for

$$\alpha\sigma \geq \gamma^2.$$

The WFP equation (1.1) is self-consistently coupled with the Poisson equation for the (real-valued) potential $V[w](x, t)$:

$$\begin{aligned} V_{xx} &= n[w] - D, & x \in (x, 2\pi), t > 0, \\ V(0, t) &= V(2\pi, t), \end{aligned}$$

with the particle density

$$n[w](x, t) = \int_{\mathbb{R}} w(x, v, t) dv. \quad (1.4)$$

$D = D(x)$ denotes the density of some fixed charges (“doping profile” in the context of semiconductor modeling), which is assumed to be given.

Mathematical properties of the Wigner-Poisson equation and dissipative Wigner systems have been intensively studied in the last decade (see [12, 3] and references therein). The (friction-free) WFPF equation in 3 dimensions was first analyzed in [4], where unique local-in-time solutions were constructed. The main analytical challenge of Wigner-Poisson systems lies in controlling the particle density (1.4) in appropriate L^p spaces. Usually this is achieved by either reformulating the Wigner equation as a Schrödinger system or a von Neumann equation ([12, 3]) or by exploiting the dissipative structure of the system ([4]). The 1-dimensional Wigner-Poisson equation, however, allows for a ‘direct’ analysis (cf. [5], §5). Hence our interest in this analytical framework.

2 EXISTENCE AND UNIQUENESS OF GLOBAL-IN-TIME SOLUTION

In this section we shall establish existence and uniqueness of global mild and classical solutions to the WFPF system (1.1)-(1.4). This system will be considered as an evolution problem in the weighted (real-valued) L^2 -space

$$X = L^2((0, 2\pi) \times \mathbb{R}; (1 + v^2)dx dv),$$

endowed with the scalar product

$$\langle u, w \rangle_X = \int_0^{2\pi} \int_{\mathbb{R}} uw(1 + v^2)dv dx.$$

This choice of the space X allows to define the particle density $n[w]$ of a Wigner function $w \in X$: a simple estimate (using Cauchy-Schwartz) yields

$$\|n[w]\|_{L^2(0, 2\pi)} \leq C\|w\|_X. \quad (2.1)$$

Here and in the sequel C denotes generic, but not necessarily equal constants.

We shall use semigroup techniques to prove existence and uniqueness of a solution to the semi-linear WFPF system (1.1)-(1.4). To this end the quadratically nonlinear potential term $\Theta[V]w$ will be considered as a bounded perturbation in the kinetic Fokker-Planck equation $w_t + vw_x = \beta(vw)_v + \sigma w_{vv} + 2\gamma w_{xv} + \alpha w_{xx}$.

We first consider the unbounded linear operator $A : D(A) \rightarrow X$,

$$Au = -v\partial_x u + \beta\partial_v(vu) + \sigma\partial_v^2 u + 2\gamma\partial_v\partial_x u + \alpha\partial_x^2 u, \quad (2.2)$$

defined on

$$D(A) = \{u \in X \mid vu_x, u_{vv}, vu_v, u_{xx}, u_{xv} \in X; u(0, v) = u(2\pi, v), \\ u_x(0, v) = u_x(2\pi, v) \forall v \in \mathbb{R}\}.$$

Clearly, the restriction (to $(0, 2\pi) \times \mathbb{R}$) of $C^\infty(\mathbb{R}^2)$ -functions that are 2π -periodic in x and have a compact support in v are included in $D(A)$. Hence, $D(A)$ is dense in X . A simple calculation shows that for $u \in D(A)$, u_v is also in X .

A straightforward calculation using the periodicity in x and integrations by part yields

$$\langle Au, w \rangle_X = \langle u, A_1^* w \rangle_X + \langle u, A_2^* w \rangle_X, \quad \forall u, w \in D(A),$$

with

$$\begin{aligned} A_1^* w &= v \partial_x w - \beta v \partial_v w + \sigma \partial_v^2 w + 2\gamma \partial_v \partial_x w + \alpha \partial_x^2 w, \\ A_2^* w &= \frac{1}{1+v^2} [2\sigma(w + 2v \partial_v w) - 2\beta v^2 w + 4\gamma v w_x]. \end{aligned}$$

Hence, $A^*|_{D(A)}$ - the restriction of the adjoint of the operator A to $D(A)$ - is given by $A^*w = A_1^*w + A_2^*w$, $w \in D(A)$. A^* is densely defined on $D(A^*) \supseteq D(A)$, and hence A is a closable operator (cf. Theorem VIII.1.b of [15]). Its closure \overline{A} satisfies $(\overline{A})^* = A^*$ (cf. [15], Theorem VIII.1.c).

Next we study the dissipation property of the operator A , which is defined on the Hilbert space X (over \mathbb{R}) by:

$$\langle Au, u \rangle_X \leq 0, \quad \forall u \in D(A).$$

Lemma 2.1 *Let the coefficients of the operator A satisfy $\alpha\sigma \geq \gamma^2$. Then $A - (\sigma + \frac{\beta}{2})I$ and its closure are dissipative.*

PROOF.- Using integrations by part we have for $u \in D(A)$:

$$\begin{aligned} \langle Au, u \rangle_X &= -\iint v u_x u + \beta \iint (vu)_v u + \sigma \iint u_{vv} u + 2\gamma \iint u_{xv} u \\ &\quad + \alpha \iint u_{xx} u - \iint v^3 u_x u + \beta \iint v^2 (vu)_v u + \sigma \iint v^2 u_{vv} u \\ &\quad + 2\gamma \iint v^2 u_{xv} u + \alpha \iint v^2 u_{xx} u \\ &= -\beta \iint v u v_v - \sigma \iint u_v u_v + 2\gamma \iint u_{xv} u - \alpha \iint u_x u_x - \beta \iint (v^2 u)_v v u \\ &\quad - \sigma \iint (v^2 u)_v u_v + 2\gamma \iint (vu)_{xv} v u - 2\gamma \iint u_x v u - \alpha \iint v^2 u_x u_x \end{aligned}$$

where $\iint f$ denotes the integral $\int_0^{2\pi} \int_{\mathbb{R}} f(x, v) dv dx$.

For the two integrals of the right side that involve mixed x - v -derivatives we shall now use the interpolation inequality

$$\iint u_{xv} u \leq \frac{\epsilon}{2} \|u_x\|_2^2 + \frac{1}{2\epsilon} \|u_v\|_2^2, \quad \epsilon > 0, \quad (2.3)$$

which is immediately obtained by an integration by parts (in v) and Young's inequality. With $\epsilon = \frac{2}{\sigma}$ we then obtain

$$\begin{aligned} \langle Au, u \rangle_X &\leq \frac{\beta}{2} \|u\|_2^2 - \sigma \|u_v\|_2^2 + \epsilon \gamma \|u_x\|_2^2 + \frac{1}{\epsilon} \gamma \|u_v\|_2^2 \\ &\quad - \alpha \|u_x\|_2^2 - 2\beta \|vu\|_2^2 - \beta \iint v^3 u_v u - 2\sigma \iint v u u_v \\ &\quad - \sigma \|v u_v\|_2^2 + \epsilon \gamma \|v u_x\|_2^2 + \frac{1}{\epsilon} \gamma \|(vu)_v\|_2^2 - \alpha \|v u_x\|_2^2 \\ &= \frac{\beta}{2} \|u\|_2^2 + \frac{\gamma^2}{\sigma} \|u_x\|_2^2 - \alpha \|u_x\|_2^2 - 2\beta \|vu\|_2^2 + \frac{3}{2} \beta \|vu\|_2^2 \\ &\quad + \sigma \|u\|_2^2 + \frac{\gamma^2}{\sigma} \|v u_x\|_2^2 - \alpha \|v u_x\|_2^2 \\ &\leq (\sigma + \frac{\beta}{2}) \|u\|_2^2, \end{aligned}$$

Thus

$$\langle [A - (\sigma + \frac{\beta}{2})I]u, u \rangle_X \leq -\sigma \|vu\|_2^2 - \frac{\beta}{2} \|vu\|_2^2 \leq 0 \quad (2.4)$$

and the operator $A - (\sigma + \frac{\beta}{2})I$ is dissipative. By Theorem 1.4.5b of [14] its closure, $\overline{A - (\sigma + \frac{\beta}{2})I} = \overline{A} - (\sigma + \frac{\beta}{2})I$ is also dissipative. \square

It is easy to see that the operator $A - \frac{\beta}{2}I$ defined on $\widetilde{D(A)} = \{u \in L^2((0, 2\pi) \times \mathbb{R}) \mid vu_x, u_{vv}, vu_v, u_{xx} \in L^2((0, 2\pi) \times \mathbb{R}); u(0, v) = u(2\pi, v), u_x(0, v) = u_x(2\pi, v), \forall v \in \mathbb{R}\}$ is dissipative in $L^2((0, 2\pi) \times \mathbb{R})$ and the L^2 -adjoint of A is $A^* = A_1^*$ on $\widetilde{D(A)}$.

Let us now study the dissipativity of the operator A^* restricted to $D(A)$. Analogously to Lemma 2.1 we have:

$$\langle A^*u, u \rangle_X \leq (\sigma + \frac{\beta}{2})\|u\|_2^2, \quad \forall u \in D(A).$$

Hence the restriction of the operator $A^* - (\sigma + \frac{\beta}{2})I = [A - (\sigma + \frac{\beta}{2})I]^*$ to $D(A)$ is dissipative.

Next we consider the dissipativity of this operator on its proper domain $D(A^*)$, which, however, is not known explicitly. To this end we shall use the following technical lemma whose proof is deferred to the appendix. Here we shall denote by \tilde{u} the (in x) 2π -periodic extension of a function $u \in X$ to \mathbb{R}^2 .

Lemma 2.2 *Let $P := p(v, \partial_x, \partial_v)$ be a linear operator in X , where p is a quadratic polynomial and*

$$D(P) := \{u \in X \mid \tilde{u} \in C^\infty(\mathbb{R}^2) \text{ with compact support in } v\} \subset X.$$

Then \overline{P} is the maximum extension of P in the sense that

$$D(\overline{P}) := \{u \in X \mid \text{the distribution } Pu \in X\}.$$

We now apply Lemma 2.2 to $P = A^* - (\sigma + \frac{\beta}{2})I$, which is dissipative on $D(P) \subset D(A)$. Since A^* is closed, we have $D(A^*) = D(\overline{P}) = \{u \in X \mid A^*u \in X\}$ and $A^* - (\sigma + \frac{\beta}{2})I$ is dissipative on all of $D(A^*)$.

Applying Corollary 1.4.4 of [14] to $\overline{A} - (\sigma + \frac{\beta}{2})I$ (with $(\overline{A})^* = A^*$), then implies that $\overline{A} - (\sigma + \frac{\beta}{2})I$ generates a C_0 semigroup of contractions on X , and the C_0 semigroup generated by \overline{A} satisfies

$$\|e^{t\overline{A}}u\|_X \leq e^{(\sigma + \frac{\beta}{2})t}\|u\|_X, \quad u \in X, t \geq 0.$$

By the same arguments $\overline{A} - \frac{\beta}{2}I$ generates a C_0 semigroup of contractions on the space $L^2((0, 2\pi) \times \mathbb{R})$.

Next we shall analyze the properties of the quadratically nonlinear term $\Theta[V]w$, which will later be considered as a perturbation of the generator \overline{A} .

For $V \in L^\infty(\mathbb{R})$ the pseudo-differential operator $\Theta[V]$ from (1.2) is defined by

$$(\mathcal{F}_v \Theta[V]u)(x, \eta) = i\delta V(x, \eta)\mathcal{F}_v u(x, \eta), \quad u \in L^2((0, 2\pi) \times \mathbb{R}_v).$$

Since $\delta V(x, \eta) \in \mathbb{R}$, the operator $\Theta[V]$ is skew-symmetric on $L^2((0, 2\pi) \times \mathbb{R}_v)$ and it satisfies

(cf. [11], [5]):

$$\|\Theta[V]\|_{\mathcal{B}(L^2((0, 2\pi) \times \mathbb{R}_v))} \leq 2\|V\|_\infty.$$

For $V \in L^\infty(\mathbb{R})$ we define the pseudo-differential operator $\Omega[V]$ on $L^2((0, 2\pi) \times \mathbb{R}_v)$ by

$$\begin{aligned} (\Omega[V]u)(x, v) &= \frac{1}{2}[V(x + \frac{1}{2i}\nabla_v) + V(x - \frac{1}{2i}\nabla_v)]u(x, v) \\ &= \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} [V(x + \frac{\eta}{2}) + V(x - \frac{\eta}{2})]\mathcal{F}_v u(x, \eta)e^{i\eta v} d\eta. \end{aligned} \quad (2.5)$$

As for the operator $\Theta[V]$ we obtain

$$\|\Omega[V]\|_{\mathcal{B}(L^2((0,2\pi)\times\mathbb{R}_v))} \leq \|V\|_\infty. \quad (2.6)$$

Proposition 2.3 *Let $V \in W^{1,\infty}(\mathbb{R})$. Then,*

$$\Theta[V](vw) = v\Theta[V]w + \Omega[V_x]w \quad (2.7)$$

holds for $w \in X$.

PROOF.- By partial integration we obtain

$$\begin{aligned} \Theta[V](vw) &= \frac{i}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} (V(x + \frac{\eta}{2}) - V(x - \frac{\eta}{2})) v' w(x, v') e^{i(v-v')\eta} dv' d\eta \\ &= \frac{i}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[(V(x + \frac{\eta}{2}) - V(x - \frac{\eta}{2})) w(x, v') e^{iv\eta} \right] \left[v' e^{-iv'\eta} \right] d\eta dv' \\ &= \frac{1}{4\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} (V_x(x + \frac{\eta}{2}) + V_x(x - \frac{\eta}{2})) w(x, v') e^{i(v-v')\eta} d\eta dv' \\ &\quad + \frac{i}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} v (V(x + \frac{\eta}{2}) - V(x - \frac{\eta}{2})) w(x, v') e^{i(v-v')\eta} d\eta dv' \\ &= \Omega[V_x]w + v\Theta[V]w. \end{aligned} \quad \square$$

Now, let us consider the nonlinear operator B defined on X by

$$u \longmapsto Bu := -\Theta[V[u]]u,$$

where $V[u]$ is the 2π -periodically extended solution of the Poisson equation

$$\begin{aligned} V_{xx} &= n[u] - D, \quad x \in (0, 2\pi), \\ V(0) &= V(2\pi), \end{aligned} \quad (2.8)$$

with $n[u](x) = \int_{\mathbb{R}} u(x, v) dv$.

Lemma 2.4 *Let $D \in L^1(0, 2\pi)$. Then*

(a) *B maps X into itself.*

(b) *Moreover, the operator B is of class C^∞ in X , and satisfies*

$$\|Bu_1 - Bu_2\|_X \leq C(\|u_1\|_X + \|u_2\|_X + \|D\|_{L^1(0,2\pi)})\|u_1 - u_2\|_X,$$

for $u_1, u_2 \in X$.

For the simple proof we refer the reader to [5].

Remark 2.5 *In the proof of Lemma 2.4 it is essential that $\|u\|_X$ controls $n[u]$ in $L^1(0, 2\pi)$ (see (2.1)). Hence the solution of the Poisson equation (2.8) satisfies $V[u] \in W^{1,\infty}(0, 2\pi)$, and $\|\Theta[V[u]]\|_{\mathcal{B}(X)} \leq C\|V[u]\|_{W^{1,\infty}(\mathbb{R})}$.*

We rewrite the WFPF system as

$$\begin{aligned} w_t &= \bar{A}w + Bw, \quad t > 0, \\ w(t=0) &= w^I \in X. \end{aligned} \quad (2.9)$$

The main result of this paper is:

Theorem 2.6 *Let $D \in L^1(0, 2\pi)$.*

a) *For every $w^I \in X$, the WPPF problem (2.9) has a unique mild solution $w \in C([0, \infty), X)$.*

b) *If $w^I \in D(\bar{A})$, w is a classical solution, i.e. $w \in C^1([0, \infty), X)$, and $w(t) \in D(\bar{A})$ for $t \geq 0$.*

PROOF.- We consider B as a bounded perturbation of the generator \bar{A} . Since B is locally Lipschitz continuous, Theorem 6.1.4 of [14] shows that (2.9) has a unique mild solution for every $w^I \in X$ on some time interval $[0, t_{max})$. Moreover, if $t_{max} = t_{max}(w^I) < \infty$ then $\lim_{t \nearrow t_{max}} \|w\|_X = \infty$. Since B is of class C^∞ in X , Theorem 6.1.5 in [14] proves that w is a classical solution on $[0, t_{max})$ for $w^I \in D(\bar{A})$.

To prove $t_{max} = \infty$ we shall now derive an a-priori estimate for $\|w(t)\|_X$.

Step 1. Here we shall derive this a-priori estimate under the assumption $w^I \in D(\bar{A})$. To this end we consider the evolution equation for $\|w\|_X^2$. By computing its time derivative and taking into account (2.9), we deduce

$$\frac{1}{2} \frac{d}{dt} \|w\|_X^2 = \langle \bar{A}w, w \rangle_X + \langle Bw, w \rangle_X .$$

Using the dissipativity of $\bar{A} - (\sigma + \frac{\beta}{2})I$ (cf. (2.4)) we conclude that

$$\frac{1}{2} \frac{d}{dt} \|w\|_X^2 \leq \left(\sigma + \frac{\beta}{2} \right) \|w\|_X^2 + \langle Bw, w \rangle_X .$$

The skew-symmetry of the operator $\Theta[V]$ implies finally that

$$\frac{1}{2} \frac{d}{dt} \|w\|_X^2 \leq \left(\sigma + \frac{\beta}{2} \right) \|w\|_X^2 + \iint vw\Omega[V_x(t)]w. \quad (2.10)$$

On the other hand, since $\bar{A} - \frac{\beta}{2}I$ is dissipative on the space $L^2((0, 2\pi) \times \mathbb{R})$, the estimates

$$\frac{d}{dt} \|w\|_2^2 \leq \beta \|w\|_2^2 \quad \text{and} \quad \|w\|_2^2 \leq \|w^I\|_2^2 e^{\beta t} \quad (2.11)$$

follow. From the proof of Lemma 2.4 in [5] we have for the solution of (2.8):

$$\|V[w]\|_{W^{1,\infty}(0,2\pi)} \leq C(\|w\|_X + \|D\|_{L^1(0,2\pi)}).$$

Using (2.6), (2.10) and (2.11) we hence obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w\|_X^2 - \left(\sigma + \frac{\beta}{2} \right) \|w\|_X^2 &\leq \iint vw\Omega[V_x(t)]w \\ &\leq \|V_x(t)\|_\infty \|vw\|_2 \|w\|_2 \\ &\leq C(\|w\|_X + \|D\|_1) \|vw\|_2 \|w^I\|_2 e^{\frac{\beta}{2}t} \\ &\leq C \|w^I\|_2 e^{\frac{\beta}{2}t} (\|w\|_X^2 + \|w\|_X \|D\|_1) \\ &\leq C \|w^I\|_2 e^{\frac{\beta}{2}t} (\|w\|_X^2 + \|D\|_1^2). \end{aligned}$$

Thus

$$\frac{d}{dt} \|w\|_X^2 \leq a(t) \|w\|_X^2 + b(t),$$

where

$$\begin{aligned} a(t) &= C \|w^I\|_2 e^{\frac{\beta}{2}t} + \beta + 2\sigma, \\ b(t) &= C \|w^I\|_2 e^{\frac{\beta}{2}t} \|D\|_1^2. \end{aligned}$$

Finally, applying Gronwall's inequality yields:

$$\|w(t)\|_X^2 \leq \|w^I\|_X^2 e^{\int_0^t a(s)ds} + \int_0^t b(s) e^{\int_s^t a(\tau)d\tau} ds, \quad t \geq 0. \quad (2.12)$$

Hence $t_{max} = \infty$ holds.

Step 2. Since (2.12) only involves $\|w^I\|_X$ this result carries over to w^I only in X by the following density argument.

For $w^I \in X$ let (w_n^I) be a sequence in $D(\overline{A})$ such that $w_n^I \rightarrow w^I$ in X . Using (2.12) we have for every w_n^I an a-priori estimate for the corresponding classical solution:

$$\|w_n(t)\|_X \leq h(t), \quad \forall t \geq 0, n \in \mathbb{N}$$

with $h \in C[0, \infty)$ independent of n .

Let $w \in C([0, t_{max}(w^I)), X)$ be the unique mild solution for w^I , which exists according to the first part of this theorem.

Next we assume $t_{max}(w^I) < \infty$. Thus $\lim_{t \nearrow t_{max}(w^I)} \|w(t)\|_X = \infty$. For the continuous, monotonously increasing function $g(t) := \max \{\|w(\tau)\|_X, 0 \leq \tau \leq t\}$ we also have $\lim_{t \nearrow t_{max}(w^I)} g(t) = \infty$.

Choose $N \in \mathbb{N}$ with $N \geq 2 \max \{h(t), t \in [0, t_{max}(w^I)]\}$. Then there exists a $t_N < t_{max}(w^I)$ such that

$$\begin{aligned} g(t_N) &= N, \\ g(t) &\leq N, \quad t \leq t_N, \\ g(t) &\geq N, \quad t_N \leq t < t_{max}(w^I). \end{aligned} \quad (2.13)$$

We denote by L_N the Lipschitz constant of the operator B on

$$B_N := \{u \in X, \|u\|_X < N\}.$$

Let \hat{B} be a (globally) Lipschitz extension of B outside of B_N . Thus, applying Theorem 6.1.2 in [14] on $[0, t_N]$ we obtain a Lipschitz dependence of the solutions on their initial values,

$$\|w - w_n\|_{C([0, t_N], X)} \leq C(L_N) \|w^I - w_n^I\|_X.$$

Thus, $w_n \rightarrow w$ in $C([0, t_N], X)$, and $\|w(t)\|_X \leq h(t) \leq \frac{N}{2}$, for $0 \leq t \leq t_N$ follows. This contradicts the assumption (2.13). \square

3 APPENDIX: PROOF OF LEMMA 2.2

To prove the assertion we shall construct for each $f \in D(\overline{P}) \subset L^2((0, 2\pi) \times \mathbb{R})$ a sequence $\{f_n\} \subset D(P)$ such that $f_n \rightarrow f$ in the graph norm $\|f\|_P = \|f\|_{L^2} + \|vf\|_{L^2} + \|Pf\|_{L^2} + \|vPf\|_{L^2}$.

To shorten the proof we shall consider here only the case

$$P = \mu + \nu v \partial_x + \beta v \partial_v + \sigma \partial_v^2 + 2\gamma \partial_v \partial_x + \alpha \partial_x^2$$

(cf. the definition of the operator A in (2.2)), but exactly the same strategy extends to the general case.

First we define the molifying delta sequence

$$\varphi_n(x, v) := n^2 \varphi(nx, nv); \quad n \in \mathbb{N}; \quad x, v \in \mathbb{R},$$

with the properties:

$$\begin{aligned}\varphi &\in C_0^\infty(\mathbb{R}^2), \\ \varphi(x, v) &\geq 0, \\ \int \int \varphi(x, v) dx dv &= 1, \\ \text{supp}\varphi &\subset \{|x|^2 + |v|^2 \leq 1\}.\end{aligned}$$

The velocity-cutoff function

$$\psi_n(v) := \psi\left(\frac{v}{n}\right); \quad n \in \mathbb{N}; \quad v \in \mathbb{R}$$

is assumed to have the properties

$$\begin{aligned}\psi &\in C_0^\infty(\mathbb{R}), \\ 0 &\leq \psi(v) \leq 1, \\ |\psi^{(j)}(v)| &\leq C_j \quad \forall v \in \mathbb{R}; \quad j = 1, 2, \\ \text{supp}\psi &\subset [-1, 1], \\ \psi|_{[-\frac{1}{2}, \frac{1}{2}]} &\equiv 1.\end{aligned}$$

We now define the approximating sequence

$$\tilde{f}_n(x, v) := (\tilde{f} * \varphi_n)(x, v) \cdot \psi_n(v), \quad n \in \mathbb{N},$$

where ‘ $*$ ’ denotes the convolution in x and v . Remember that \tilde{f} denotes the (in x) 2π -periodic extension of the function $f \in X$ to \mathbb{R}^2 . By construction we have $\tilde{f}_n \in C^\infty(\mathbb{R}^2)$ and \tilde{f}_n is 2π -periodic in x with compact support in v . Now, let R denote the restriction operator of (in x) 2π -periodic functions to $(0, 2\pi) \times \mathbb{R}$. Then, $f_n := R\tilde{f}_n \in D(P)$. According to the 4 terms of the graph norm we split the proof into 4 steps:

Step 1: Since $\varphi_n \rightarrow \delta$ in $\mathcal{D}'(\mathbb{R}^2)$ and $\psi_n(v) \rightarrow 1$ pointwise, we have $\tilde{f}_n \rightarrow \tilde{f}$ in $L^2_{loc}(\mathbb{R}_x) \times L^2(\mathbb{R}_v)$ and

$$f_n \rightarrow f \quad \text{in } L^2((0, 2\pi) \times \mathbb{R}).$$

Step 2: For the second term of the graph norm we write

$$v\tilde{f}_n = (v\tilde{f} * \varphi_n)\psi_n + (\tilde{f} * v\varphi_n)\psi_n.$$

The restriction of the first summand converges to vf in $L^2((0, 2\pi) \times \mathbb{R})$ and the second term converges to 0 since $v\varphi_n \rightarrow 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Hence we have

$$f_n \rightarrow f \quad \text{in } X.$$

Step 3: To prove that $Pf_n \rightarrow Pf$ in $L^2((0, 2\pi) \times \mathbb{R})$ we write:

$$\begin{aligned}P\tilde{f}_n &= \mu(\tilde{f} * \varphi_n)\psi_n + \nu(v\tilde{f}_x * \varphi_n)\psi_n + \beta(v\tilde{f}_v * \varphi_n)\psi_n \\ &\quad + \sigma(\tilde{f}_{vv} * \varphi_n)\psi_n + 2\gamma(\tilde{f}_{xv} * \varphi_n)\psi_n + \alpha(\tilde{f}_{xx} * \varphi_n)\psi_n \\ &\quad + r_n^1(x, v) \\ &= (P\tilde{f} * \varphi_n)\psi_n + r_n^1(x, v).\end{aligned}$$

As we shall show, the restriction of all six terms of the remainder

$$\begin{aligned} r_n^1 &= \nu(\tilde{f} * v\partial_x\varphi_n)\psi_n + \beta(\tilde{f} * \varphi_n)(v\partial_v\psi_n) \\ &\quad + \beta(\tilde{f} * \partial_v(v\varphi_n))\psi_n + 2\sigma(\tilde{f} * (\frac{1}{n}\partial_v\varphi_n))(n\partial_v\psi_n) \\ &\quad + \sigma(\tilde{f} * \varphi_n)\partial_v^2\psi_n + 2\gamma(\tilde{f} * (\frac{1}{n}\partial_x\varphi_n))(n\partial_v\psi_n) \end{aligned}$$

converge to 0 in $L^2((0, 2\pi) \times \mathbb{R})$:

In the first term $v\partial_x\varphi_n \rightarrow 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Hence we have

$$R\left(\tilde{f} * v\partial_x\varphi_n\right) \rightarrow 0 \quad \text{in } L^2((0, 2\pi) \times \mathbb{R}),$$

and the same argument holds for the third term.

For the second term we have

$$v\partial_v\psi_n = \frac{v}{n}\psi'\left(\frac{v}{n}\right),$$

which is in $L^\infty(\mathbb{R})$, uniformly for $n \in \mathbb{N}$ and with support in $[-n, -\frac{n}{2}] \cup [\frac{n}{2}, n]$. Hence, the second term converges to 0 in $L^2((0, 2\pi) \times \mathbb{R})$.

In the fourth term $\frac{1}{n}\partial_v\varphi_n \rightarrow 0$ in $\mathcal{D}'(\mathbb{R}^2)$, and hence

$$R\left(\tilde{f} * \left(\frac{1}{n}\partial_v\varphi_n\right)\right) \rightarrow 0 \quad \text{in } L^2((0, 2\pi) \times \mathbb{R}).$$

Furthermore, $n\partial_v\psi_n = \psi'\left(\frac{v}{n}\right)$ with $|\psi'| \leq C_1$. By the same argument also the sixth term converges to 0 in $L^2((0, 2\pi) \times \mathbb{R})$.

Finally, the fifth term converges to 0 since $\partial_v^2\psi_n = \frac{1}{n^2}\psi''\left(\frac{v}{n}\right)$ with $|\psi''| \leq C_2$.

Step 4: To prove that $vPf_n \rightarrow vPf$ in $L^2((0, 2\pi) \times \mathbb{R})$ we write:

$$\begin{aligned} vP\tilde{f}_n &= \mu(v\tilde{f} * \varphi_n)\psi_n + \nu(v^2\tilde{f}_x * \varphi_n)\psi_n + \beta(v^2\tilde{f}_v * \varphi_n)\psi_n \\ &\quad + \sigma(v\tilde{f}_{vv} * \varphi_n)\psi_n + 2\gamma(v\tilde{f}_{xv} * \varphi_n)\psi_n + \alpha(v\tilde{f}_{xx} * \varphi_n)\psi_n \\ &\quad + r_n^2(x, v) \\ &= ((vP\tilde{f}) * \varphi_n)\psi_n + r_n^2(x, v), \end{aligned}$$

with the remainder

$$\begin{aligned} r_n^2 &= \mu(\tilde{f} * v\varphi_n)\psi_n + 2\nu(v\tilde{f} * v\partial_x\varphi_n)\psi_n + \nu(\tilde{f} * v^2\partial_x\varphi_n)\psi_n \\ &\quad + \beta(v\tilde{f} * \varphi_n + \tilde{f} * v\varphi_n)v\partial_v\psi_n + 2\beta(v\tilde{f} * \partial_v(v\varphi_n))\psi_n \\ &\quad + \beta(\tilde{f} * v^2\partial_v\varphi_n)\psi_n + \sigma(\tilde{f} * \partial_{vv}(v\varphi_n))\psi_n \\ &\quad + 2\sigma\left(v\tilde{f} * \frac{\partial_v\varphi_n}{n} + \tilde{f} * \frac{v\partial_v\varphi_n}{n}\right)\psi'\left(\frac{v}{n}\right) \\ &\quad + \sigma(\tilde{f} * \varphi_n)v\partial_v^2\psi_n + 2\gamma(\tilde{f} * \partial_{xv}(v\varphi_n))\psi_n \\ &\quad + 2\gamma\left(v\tilde{f} * \frac{\partial_x\varphi_n}{n} + \tilde{f} * \frac{v\partial_x\varphi_n}{n}\right)\psi'\left(\frac{v}{n}\right) + \alpha(\tilde{f} * v\partial_{xx}\varphi_n)\psi_n. \end{aligned}$$

For proving that the restriction of all terms of r_n^2 converge to 0 in $L^2((0, 2\pi) \times \mathbb{R})$ we recall that both $f, vf \in L^2((0, 2\pi) \times \mathbb{R})$. Since the strategy of the proof is the same as in Step 3 we shall only give the key points:

The distributions $v\varphi_n, v\partial_x\varphi_n, v^2\partial_x\varphi_n, \partial_v(v\varphi_n), v^2\partial_v\varphi_n, \partial_{vv}(v\varphi_n), \frac{\partial_v\varphi_n}{n}, \frac{v\partial_v\varphi_n}{n}, \partial_{xv}(v\varphi_n), \frac{\partial_x\varphi_n}{n}, \frac{v\partial_x\varphi_n}{n}$, and $v\partial_{xx}\varphi_n$ all converge to 0 in $\mathcal{D}'(\mathbb{R}^2)$. Further, $v\partial_v^2\psi_n \rightarrow 0$ in $L^\infty(\mathbb{R})$ and the term $v\partial_v\psi_n$ was already discussed in Step 3. \square

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References

- [1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
- [2] H.W. Alt, *Lineare Funktionalanalysis*, Springer, Berlin, 1985
- [3] A. Arnold *Self-consistent relaxation-time models in quantum mechanics*, Comm. PDE, 21(3&4):473–506, 1995.
- [4] A. Arnold, J.L. Lopez, P.A. Markowich, J. Soler, *An Analysis of Quantum Fokker-Planck Models: A Wigner Function Approach*, submitted to Rev. Mat. Iberoam., 2001.
- [5] A. Arnold, C. Ringhofer, *An operator splitting method for the Wigner-Poisson problem*, SIAM J.Numer.Anal., 33:1622–1643, 1996.
- [6] A.O. Caldeira and A.J. Leggett, *Path integral approach to quantum Brownian motion*, Physica A, 121:587–616, 1983.
- [7] J. A. Carrillo, J. Soler, J. L. Vázquez, *Asymptotic behaviour and selfsimilarity for the three dimensional Vlasov-Poisson-Fokker-Planck system*, J. Funct. Anal. 141:99-132, 1996.
- [8] L. Diósi, *On high-temperature Markovian equation for quantum Brownian motion*, Europhys. Lett., 22:1–3, 1993.
- [9] H.L. Gruvin, T.R. Govindan, J.P. Kreskovsky, and M.A. Stroschio, *Transport via the Liouville equation and moments of quantum distribution functions*, Solid State Electr., 36:1697–1709, 1993.
- [10] E. Kreyszig, *Introductory functional analysis with applications*, John Wiley, New York, 1978.
- [11] P.A. Markowich, C. Ringhofer, *An analysis of the quantum Liouville equation*, Z.Angew.Math.Mech., 69:121-127, 1989.
- [12] P.A. Markowich, C. Ringhofer, C. Schmeiser, *Semiconductor Equations*, Springer, New York, 1990.
- [13] A.C. McBride, *Semigroups of linear Operators. An introduction*, Longman Scientific & Technical, Harlow, Essex, UK, 1987.
- [14] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, Berlin/New York, 1983.
- [15] M. Reed & B. Simon, *Methods of Modern Mathematical Physics I, Functional Analysis*, Academic Press, London, 1980.
- [16] M. Reed & B. Simon, *Methods of Modern Mathematical Physics II, Fourier Analysis, Self-Adjointness*, Academic Press, London, 1975.
- [17] H. Risken, *The Fokker-Planck Equation*, Springer Berlin-Heidelberg, 1984.
- [18] M.A. Stroschio, *Moment-equation representation of the dissipative quantum Liouville equation*, Superlattices and Microstructures, 2:83–87, 1986.
- [19] E. Wigner, *On the quantum correction for thermodynamic equilibrium*, Phys. Rev. 40:749–759, 1932.