Refined Convex Sobolev Inequalities

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Abstract

This paper is devoted to refinements of convex Sobolev inequalities in the case of power law relative entropies: a nonlinear entropy–entropy production relation improves the known inequalities of this type. The corresponding generalized Poincaré type inequalities with weights are derived. Optimal constants are compared to the usual Poincaré constant.

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1 Introduction and main results

In this paper, we consider convex Sobolev inequalities relating a (non-negative) convex entropy functional

$$e_\psi(\rho|\rho_\infty) := \int_{\mathbb{R}^n} \psi\Big(\frac{\rho}{\rho_\infty}\Big)\, d\rho_\infty$$

to an entropy production functional

$$I_\psi(\rho|\rho_\infty) := -\int_{\mathbb{R}^n} \psi''\Big(\frac{\rho}{\rho_\infty}\Big) D\left|\nabla\Big(\frac{\rho}{\rho_\infty}\Big)\right|^2 d\rho_\infty\ , \tag{1.1}$$

where ρ and ρ_∞ belong to $L^1_+(\mathbb{R}^n, dx)$ and satisfy $\|\rho\|_{L^1(\mathbb{R}^n)} = \|\rho_\infty\|_{L^1(\mathbb{R}^n)} = M > 0$. Here we use the notation $d\rho_\infty = \rho_\infty(x)\ dx$. The *generating function* $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ of the relative entropy is strictly convex and satisfies $\psi(1) = 0$.

A very efficient method to prove convex Sobolev inequalities has been developped by D. Bakry and M. Emery [3, 4] in probability theory and by A. Arnold, P. Markowich, G. Toscani, A. Unterreiter [2] in the context of partial differential equations. See [1] for a recent review. The main idea goes as follows: We consider $\rho = \rho(x,t)$ depending now on the auxiliary variable $t > 0$ ("time"). For any solution of

$$\frac{\partial \rho}{\partial t} = \operatorname{div}\left(D\,\rho_\infty \nabla\Big(\frac{\rho}{\rho_\infty}\Big)\right), \quad x \in \mathbb{R}^n,\ t > 0\,, \tag{1.2}$$

the time evolution of the relative entropy is given by the entropy production:

$$\frac{d}{dt} e_\psi(\rho(t)|\rho_\infty) = I_\psi(\rho(t)|\rho_\infty) \leq 0\,.$$

In (1.1) and (1.2) $D = D(x)$ denotes a (positive) scalar diffusion coefficient, and we assume $D \in W^{2,\infty}_{loc}(\mathbb{R}^n)$. It is also clear that $\rho_\infty(x)$ is a steady state solution of (1.2).

For $D \equiv 1$, the main assumption is that $A := -\log \rho_\infty$ is a uniformly convex function, *i.e.*

(A1)

$$\lambda_1 := \inf_{\substack{x\in\mathbb{R}^n\\ \xi\in S^{n-1}}} \left(\xi, \frac{\partial^2 A}{\partial x^2}(x)\,\xi\right) > 0\,.$$

For $D \not\equiv 1$ the corresponding assumption reads:

(A2) $\exists\,\lambda_1 > 0$ such that for any $x \in \mathbb{R}^n$

$$\Big(\frac{1}{2} - \frac{n}{4}\Big)\frac{1}{D}\nabla D \otimes \nabla D\ + \frac{1}{2}(\Delta D - \nabla D \cdot \nabla A)\mathbb{I} \\ + D\frac{\partial^2 A}{\partial x^2} + \frac{1}{2}\Big(\nabla A \otimes \nabla D + \nabla D \otimes \nabla A\Big) - \frac{\partial^2 D}{\partial x^2} \geq \lambda_1 \mathbb{I}$$

(in the sense of positive definite matrices). Here $\mathbb{I}$ denotes the identity matrix. In these two cases, one can prove the *convex Sobolev inequality*

$$e_\psi(\rho|\rho_\infty) \leq \frac{1}{2\lambda_1}|I_\psi(\rho|\rho_\infty)| \quad \forall \rho \in L^1_+(\mathbb{R}^n) \text{ with } \|\rho\|_{L^1(\mathbb{R}^n)} = M \tag{1.3}$$

by computing

$$R_\psi(\rho(t)|\rho_\infty) := \frac{d}{dt}\Big[I_\psi(\rho(t)|\rho_\infty) + 2\lambda_1 e_\psi(\rho(t)|\rho_\infty)\Big]$$

and proving that

$$R_\psi(\rho(t)|\rho_\infty) \geq 0\,. \tag{1.4}$$

Integrating this differential inequality from t to ∞ then yields (1.3).

Actually, these calculations can only be carried out only for *admissible relative entropies* where $\psi \in C^4(\mathbb{R}^+)$ has to satisfy

$$2\,(\psi''')^2 \leq \psi''\,\psi^{IV} \quad \text{on } \mathbb{R}^+.$$

Typical and the most important – for practical applications – examples are generating functions of the form

$$\psi_p(\sigma) = \sigma^p - 1 - p\,(\sigma - 1) \quad \text{for } p \in (1, 2]\,, \tag{1.5}$$

and

$$\tilde{\psi}_1(\sigma) = \sigma \log \sigma - \sigma + 1\,,$$

which corresponds to the limiting case of $\psi_p/(p-1)$ as $p \to 1$. With $\psi = \tilde{\psi}_1$, Inequality (1.3) is exactly the *logarithmic Sobolev inequality* found by L. Gross [8, 9], and generalized by many authors later on.

Analyzing the precise form of $R_\psi(\rho|\rho_\infty)$ allows us to identify cases of optimality of (1.3) under the assumption $D \equiv 1$. For $p = 1$ or 2, and for potentials A that are quadratic in at least one coordinate direction (with convexity λ_1) there exist *extremal functions* $\rho = \rho_{ex} \neq \rho_\infty$ such that (1.3) becomes an equality, *cf.* [2]. Some of these optimality results were already noted by E. Carlen [6], M. Ledoux [12], and G. Toscani [14].

The non-optimality of the other cases may have two reasons: either λ_1 from (A1), (A2) is not the sharp *convex Sobolev constant* (an example for this is $A(x) = x^4$, $x \in \mathbb{R}$: see §3.3 of [2]), or there exists no extremal function to saturate (1.3), even for the sharp constant λ_1. This happens for the entropies with $p \in (1, 2)$, and it is due to the fact that the linear relationship of $|I_\psi|$ and e_ψ is then not optimal.

A refinement of (1.3) for $p \in (1, 2)$ is the topic of this paper. In this case, the non-optimality of (1.3) stems from the fact that, for any fixed D and ρ_∞,

$$J(e, e', M) := \inf_{\substack{I_\psi(\rho|\rho_\infty) = e',\ e_\psi(\rho|\rho_\infty) = e \\ \rho \in L^1_+(\mathbb{R}^n),\ \|\rho\|_{L^1(\mathbb{R}^n) = M}}} R_\psi(\rho|\rho_\infty)$$

is a positive quantity for $e > 0$ and $e' \leq -2\lambda_1 e$. Here, the t-derivatives entering in R_ψ are defined via (1.2). Our main result is based on a lower bound for $J(e, e', M)$:

$$J(e, e', M) \geq \frac{2-p}{p} \cdot \frac{|e'|^2}{M+e} \,,$$

which yields an improvement of (1.3). Finding the minimizers of J (if they exist) is probably difficult.

Theorem 1 *Let ρ_∞ satisfy (A2) for some $\lambda_1 > 0$, and take $\psi = \psi_p$ for some $p \in (1, 2)$. Then*

$$k(e) = k\left(\int_{\mathbb{R}^n} \psi\Big(\frac{\rho}{\rho_\infty}\Big)\, d\rho_\infty\right) \leq \frac{1}{2\lambda_1} \int_{\mathbb{R}^n} \psi''\Big(\frac{\rho}{\rho_\infty}\Big) D \left|\nabla\Big(\frac{\rho}{\rho_\infty}\Big)\right|^2 d\rho_\infty = \frac{1}{2\lambda_1}|I| \quad (1.6)$$

holds for any $\rho \in L^1_+(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \rho\, dx = \int_{\mathbb{R}^n} \rho_\infty\, dx = M$, where

$$k(e) := \frac{M}{1-\kappa}\left(1 + \frac{e}{M} - \Big(1 + \frac{e}{M}\Big)^\kappa\right), \quad \kappa = \frac{2-p}{p} \,.$$

We will show that there are still no extremal functions to saturate the *refined convex Sobolev inequality* (1.6). Therefore it is not yet known whether the above functional dependence of $|I_\psi|$ and e_ψ is optimal. But it improves upon (1.3) since we have

$$k(e) > e \,, \quad \forall\, e > 0 \,, \quad (1.7)$$

and the best possible constants λ_1 are shown to be independent of p (see Theorem 4).

Also, the presented method can be extended to the case $\lambda_1 = 0$ (see Proposition 3 below), thus giving a decay rate of $t \mapsto I_\psi(\rho(t)|\rho_\infty)$ for any solution ρ of (1.2), even if A is not uniformly convex. We remark that nonlinear entropy–entropy production inequalities, or "defective logarithmic Sobolev inequalities," have been derived for the logarithmic entropy (*i.e.* $\psi = \tilde{\psi}_1$) and Gaussian measures ρ_∞ (*cf.* §1.3, §4.3 of [13]).

Next we consider reformulations of the convex Sobolev inequalities (1.3) and (1.6). We assume $M = 1$ and substitute

$$\frac{\rho}{\rho_\infty} = \frac{|f|^{\frac{2}{p}}}{\int_{\mathbb{R}^n} |f|^{\frac{2}{p}}\, d\rho_\infty} \tag{1.8}$$

in (1.3) to obtain the generalized Poincaré inequalities derived by W. Beckner for Gaussian measures ρ_∞ in [5] and generalized in [2] for log-convex measures:

$$\frac{p}{p-1}\left[\int_{\mathbb{R}^n} f^2\, d\rho_\infty - \left(\int_{\mathbb{R}^n} |f|^{2/p}\, d\rho_\infty\right)^p\right] \le \frac{2}{\lambda_1}\int_{\mathbb{R}^n} D|\nabla f|^2\, d\rho_\infty \tag{1.9}$$

for all $f \in L^{2/p}(d\rho_\infty)$, $1 < p \le 2$. In the limit $p \to 1$ this yields the logarithmic Sobolev inequality:

$$\int_{\mathbb{R}^n} f^2 \log\left(\frac{f^2}{\|f\|^2_{L^2(d\rho_\infty)}}\right) d\rho_\infty \le \frac{2}{\lambda_1}\int_{\mathbb{R}^n} D|\nabla f|^2\, d\rho_\infty \tag{1.10}$$

for all $f \in L^2(d\rho_\infty)$. Hence, (1.9) interpolates between the (classical) Poincaré and the logarithmic Sobolev inequalities. A discussion on the interplay between (1.9), (1.10) and additional inequalities "between Poincaré and log. Sobolev" can be found in [11] and in §3 below. In [12] such interpolation inequalities are discussed for the Ornstein–Uhlenbeck process on $\mathbb{R}^n$ and for the heat semigroup on spheres.

Using the transformation (1.8) on the refined Sobolev inequality (1.6) directly yields a refinement of (1.9), which is nothing else than a reformulation of (1.6):

Theorem 2 *Let ρ_∞ satisfy (A2) for some $\lambda_1 > 0$ and assume that $\int_{\mathbb{R}^n} d\rho_\infty = 1$. Then*

$$\frac{1}{2}\left(\frac{p}{p-1}\right)^2\left[\int_{\mathbb{R}^n} f^2\, d\rho_\infty - \left(\int_{\mathbb{R}^n} |f|^{2/p}\, d\rho_\infty\right)^{2(p-1)} \cdot \left(\int_{\mathbb{R}^n} f^2\, d\rho_\infty\right)^{\frac{2}{p}-1}\right] \le \frac{2}{\lambda_1}\int_{\mathbb{R}^n} D|\nabla f|^2\, d\rho_\infty \tag{1.11}$$

holds for all $f \in L^{2/p}(d\rho_\infty)$, $1 < p \le 2$ and the limit $p \to 1$ again yields (1.10).

Note that the left hand sides of (1.9) and (1.11) are related by

$$\frac{1}{2}\left(\frac{p}{p-1}\right)^2\left[\int_{\mathbb{R}^n} f^2\, d\rho_\infty - \left(\int_{\mathbb{R}^n} |f|^{2/p}\, d\rho_\infty\right)^{2(p-1)} \cdot \left(\int_{\mathbb{R}^n} f^2\, d\rho_\infty\right)^{\frac{2}{p}-1}\right] \ge \frac{p}{p-1}\left[\int_{\mathbb{R}^n} f^2\, d\rho_\infty - \left(\int_{\mathbb{R}^n} |f|^{2/p}\, d\rho_\infty\right)^p\right] \tag{1.12}$$

as a consequence of (1.7) and (1.8). This can of course be recovered using Hölder's inequality:

$$\left(\int_{\mathbb{R}^n} |f|^{2/p}\, d\rho_\infty\right)^p \le \int_{\mathbb{R}^n} |f|^2\, d\rho_\infty \tag{1.13}$$

and the inequality: $\frac{1}{2}\frac{p}{p-1}(1-t^{\frac{2}{p}(p-1)}) \geq 1-t$ for any $t \in [0,1]$, $p \in (1,2]$. Note that the equality holds in (1.12) if and only if $1 = t = \left(\int_{\mathbb{R}^n} |f|^{2/p}\, d\rho_\infty\right)^p \left(\int_{\mathbb{R}^n} |f|^2\, d\rho_\infty\right)^{-1}$, *i.e.* if f is a constant.

For $p = 2$, (1.9) and (1.11) hold without absolute values, *cf.* [2], provided $\psi_2(\sigma) = \sigma^2 - 1 - 2(\sigma - 1)$ is defined over the whole real line. In that case, ρ is allowed to take negative values.

In the next section, we shall prove Theorems 1 and 2 and exploit the method in the case $\lambda_1 = 0$. Further results on best constants, perturbations and connections with Poincaré inequalities are presented in Section 3.

2 Convex Sobolev inequalities for power law entropies

Here and in the sequel we shall assume for simplicity that

$$\int_{\mathbb{R}^n} \rho_\infty\, dx = M = 1\ .$$

The general case $M > 0$ then immediately follows by scaling.

Proof of Theorem 1. Since the first part of the proof is identical to §2.3 of [2] we shall not go into details here. After a sequence of integrations by parts, dI_ψ/dt can be written as

$$\begin{aligned}
&\frac{d}{dt} I_\psi(\rho(t)|\rho_\infty) \\
&= 2\int_{\mathbb{R}^n} \psi''(\mu) D \Bigg[u^\top \nabla \otimes (\nabla A D - \nabla D) u \\
&\quad + \frac{1}{2}\,\Delta\, D|u|^2 - \frac{1}{2}|u|^2 \nabla D \cdot \nabla A + \frac{1}{D}\frac{2-n}{4}(u\cdot\nabla D)^2 \Bigg] d\rho_\infty \qquad (2.1)\\
&\quad + \int_{\mathbb{R}^n} \Bigg[\psi^{\rm IV}(\mu) D^2 |u|^4 + \psi'''(\mu)(4D^2 u^\top \frac{\partial u}{\partial x} u + 2D|u|^2 \nabla\mu\cdot\nabla D) \\
&\quad + 2\,\psi''(\mu) \sum_{i,j} \left(D\frac{\partial^2 \mu}{\partial x_i \partial x_j} + \frac{1}{2}\frac{\partial D}{\partial x_i}\frac{\partial \mu}{\partial x_j} + \frac{1}{2}\frac{\partial \mu}{\partial x_i}\frac{\partial D}{\partial x_j} - \frac{1}{2}\delta_{ij}\nabla D\cdot\nabla\mu \right)^2 \Bigg] d\rho_\infty\ ,
\end{aligned}$$

where we used the notation $\mu = \frac{\rho}{\rho_\infty}$ and $u = \nabla\mu$. Using (A2), the first integral of (2.1) can be estimated below by $-2\lambda_1 I_\psi(\rho(t)|\rho_\infty)$. In the second integral, we now insert ψ_p from (1.5) and write it as a sum of squares. This is the key step in our analysis, where we deviate from the strategy of [2] by using a sharper estimate:

$$\begin{aligned}
\frac{d}{dt} I_\psi(\rho(t)|\rho_\infty) \;\geq\; & -2\lambda_1\, I_\psi(\rho(t)|\rho_\infty) + p(p-1)^2(2-p) \int_{\mathbb{R}^n} \mu^{p-4} D^2 |u|^4\, d\rho_\infty \\
& + 2p(p-1) \int_{\mathbb{R}^n} \mu^{p-2} \sum_{i,j} \Bigg(\frac{p-2}{\mu} D \frac{\partial\mu}{\partial x_i}\frac{\partial\mu}{\partial x_j} \qquad (2.2)\\
& + D\frac{\partial^2\mu}{\partial x_i \partial x_j} + \frac{1}{2}\frac{\partial D}{\partial x_i}\frac{\partial \mu}{\partial x_j} + \frac{1}{2}\frac{\partial \mu}{\partial x_i}\frac{\partial D}{\partial x_j} - \frac{1}{2}\delta_{ij}\nabla D\cdot\nabla\mu \Bigg)^2 d\rho_\infty \\
\geq\; & -2\lambda_1\, I_\psi(\rho(t)|\rho_\infty) + p(p-1)^2(2-p) \int_{\mathbb{R}^n} \mu^{p-4} D^2 |u|^4\, d\rho_\infty\ .
\end{aligned}$$

In the two limiting cases $p = 1$ (replace ψ_p by $\tilde{\psi}_1$) and $p = 2$, the second term on the r.h.s. of (2.2) disappears. In [2], this term was always disregarded. For $1 < p < 2$, however, it makes it possible to improve (1.3).

Using the Cauchy-Schwarz inequality, we have the estimate

$$\left(\int_{\mathbb{R}^n} \mu^{p-2} D|u|^2 \, d\rho_\infty\right)^2 \le \int_{\mathbb{R}^n} \mu^{p-4} D^2|u|^4 \, d\rho_\infty \cdot \int_{\mathbb{R}^n} \mu^p \, d\rho_\infty$$
$$= \int_{\mathbb{R}^n} \mu^{p-4} D^2|u|^4 \, d\rho_\infty \cdot \int_{\mathbb{R}^n} [\psi(\mu) + 1] \, d\rho_\infty$$

and hence

$$\int_{\mathbb{R}^n} \mu^{p-4} D^2|u|^4 \, d\rho_\infty \ge \left(\frac{I_\psi(\rho|\rho_\infty)}{p(p-1)}\right)^2 \cdot [e_\psi(\rho|\rho_\infty) + 1)]^{-1} \ .$$

With the notation $e(t) = e_\psi(\rho(t)|\rho_\infty)$, we get from (2.2)

$$e'' \ge -2\lambda_1 e' + \kappa \frac{|e'|^2}{1+e} \ . \tag{2.3}$$

From (2.3) we shall now derive

$$|e'| = -e' \ge 2\lambda_1 \, k(e) \ , \tag{2.4}$$

which is the assertion of Theorem 1. We first note that both $I_\psi(\rho(t)|\rho_\infty)$ and $e_\psi(\rho(t)|\rho_\infty)$ decay exponentially with the rate $-2\lambda_1$. This follows, respectively, from (1.4) and from the usual convex Sobolev inequality (1.3).

The function

$$k(e) = \frac{1}{1-\kappa}\Big(1 + e - (1+e)^\kappa\Big)$$

is the solution of

$$k' = 1 + \kappa \frac{k(e)}{1+e} \ , \quad k(0) = 0 \ .$$

Let

$$y(t) = \big[e'(t) + 2\lambda_1 \, k(e(t))\big] \cdot e^{-\kappa \int_0^t \frac{e'(s)}{1+e(s)} \, ds} \ .$$

For any $t \ge 0$, we calculate

$$y'(t) = \left(e''(t) + 2\lambda_1 e'(t) - \kappa \frac{|e'(t)|^2}{1+e(t)}\right) \cdot e^{-\kappa \int_0^t \frac{e'(s)}{1+e(s)} \, ds} \ .$$

Since

$$|y(t)| \le |e'(t) + 2\lambda_1 \, k(e(t))| \cdot e^{-\kappa \int_0^t e'(s) \, ds} = |e'(t) + 2\lambda_1 \, k(e(t))| e^{-\kappa[e(t)-e(0)]} \to 0$$

as $t \to +\infty$, we conclude that $y(t) \le 0$, which proves (2.4). □

As we had to expect, one recovers the usual convex Sobolev inequality (1.3) in the limiting cases $p = 1$ (take the limit $p \to 1$ after dividing (1.6) by $p - 1$) and $p = 2$ (this gives $\kappa = 0$ and $k(e) = e$).

For $1 < p < 2$, we notice that $\frac{1}{1-\kappa} \, e > k(e) > e$ for any $e > 0$, but

$$\lim_{e \to 0_+} \frac{k(e)}{e} = 1 \ .$$

Hence, the estimate of Theorem 1 does not improve the asymptotic convergence rate of the solution of Equation (1.2) except for $\lambda_1 = 0$:

Proposition 3 *With the above notations, let* $\lambda_1 = 0$ *and* $1 < p < 2$*. Any solution of Equation (1.2) satisfies*

$$|I_\psi(\rho(t)|\rho_\infty)| \leq \frac{I_0}{1+\alpha t} \quad \forall\, t > 0$$

with $I_0 = |I_\psi(\rho(0)|\rho_\infty)|$ *and* $\alpha = \kappa\,\frac{I_0}{1+e_\psi(\rho(0)|\rho_\infty)}$.

Proof. Inequality (2.3) can be rewritten in the form

$$-\frac{|e'|'}{|e'|^2} \geq \frac{\kappa}{1+e} \geq \frac{\kappa}{1+e(0)}\ ,$$

thus proving the result. □

Next we address the question of saturation of the refined convex Sobolev inequality (1.6), for simplicity only for the case $D \equiv 1$. Using the strategy from [2] we rewrite (2.1) as

$$e'' = -2\lambda_1 e' + \kappa\frac{|e'|^2}{1+e} + r_\psi(\rho(t))\ ,$$

where the remainder term is

$$\begin{aligned} & r_\psi(\rho(t)) \\ &= 2\int_{\mathbb{R}^n} \psi''(\mu) u^\top \left(\frac{\partial^2 A}{\partial x^2} - \lambda_1 \mathbb{I}\right) u \, d\rho_\infty \\ &\quad + 2\,p(p-1)\int_{\mathbb{R}^n} \mu^{2-p} \sum_{i,j} \left(\frac{\partial z_i}{\partial x_j}\right)^2 d\rho_\infty \\ &\quad + \frac{p(p-1)^2(2-p)}{e+1} \cdot \left[\int_{\mathbb{R}^n} \mu^p d\rho_\infty \cdot \int_{\mathbb{R}^n} \mu^{p-4}|u|^4 d\rho_\infty - \left(\int_{\mathbb{R}^n} \mu^{p-2}|u|^2 d\rho_\infty\right)^2\right] \geq 0\ , \end{aligned} \tag{2.5}$$

with the notation $z = \mu^{p-2}\nabla\mu$. Using the notation from the proof of Theorem 1, we have

$$y'(t) = r_\psi(\rho(t))\ e^{-\kappa\int_0^t \frac{e'(s)}{1+e(s)}\,ds}\ ,$$

and an integration with respect to t gives

$$-y(0) = |e'(0)| - \lambda_1\, k(e(0)) = \int_0^\infty r_\psi(\rho(t))\ e^{-\kappa\int_0^t \frac{e'(s)}{1+e(s)}\,ds}\ dt \geq 0\ .$$

Hence we conclude that (2.4) becomes an equality, for $\rho = \rho(0)$, if and only if the remainder vanishes along the whole trajectory of $\rho(t)$, *i.e.*

$$r_\psi(\rho(t)) = 0\ , \quad t \in \mathbb{R}^+ \text{ a.e.}$$

However, no extremal function can simultaneously annihilate the second integral and the square bracket of (2.5): to make the second integral vanish, the function μ has to be of the form $\mu(x) = (C_1 + C_2 \cdot x)^{\frac{1}{p-1}}$ (whenever $\mu(x) \neq 0$), and for the last term it would have to be $\mu(x) = e^{C_1 + C_2 \cdot x}$. Hence, (2.4) does not admit extremal functions.

3 Further results and comments

In the previous sections we derived convex Sobolev inequalities (corresponding to power law entropies) for steady state measures $\rho_\infty = e^{-A(x)}$, whose potential $A(x)$ satisfies the *Bakry–Emery condition* (A2). However, such inequalities hold also in much more general situations: As soon as ρ_∞ gives rise to a (classical) Poincaré inequality (*cf.* (3.1) below), convex Sobolev inequalities of type (1.3), (1.6), (1.9), and (1.11) hold for $p \in (1,2]$. Note that this condition is much weaker than the assumption (A2).

3.1 Spectral gap, Poincaré and convex Sobolev inequalities

Using the Poincaré constant

$$\Lambda_2 \quad := \quad \inf_{\substack{w \in \mathcal{D}(\mathbb{R}^n) \\ w \not\equiv 0,\ \int_{\mathbb{R}^n} w\, d\rho_\infty = 0}} \frac{\int_{\mathbb{R}^n} D|\nabla w|^2\, d\rho_\infty}{\int_{\mathbb{R}^n} |w|^2\, d\rho_\infty} \tag{3.1}$$

we shall now give an estimate on the sharp constant in the refined Sobolev inequality (1.6) and its reformulation (1.11):

Theorem 4 *Let* $D = D(x) > 0$ *and assume that* $\rho_\infty \in L^1_+(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n} \rho_\infty\, dx =: M = 1$ *is such that* $\Lambda_2 > 0$. *Then,* refined convex Sobolev inequalites *of type (1.6) hold for any* $p \in (1,2]$. *And the optimal constant*

$$\Lambda_p := \frac{1}{2} \inf_{\substack{\rho_\infty \not\equiv \rho \in L^1_+(\mathbb{R}^n) \\ \int_{\mathbb{R}^n} \rho\, dx = M}} \frac{\left| I_{\psi_p}(\rho|\rho_\infty)\right|}{k\left(e_{\psi_p}(\rho|\rho_\infty)\right)}$$

satisfies the estimate

$$4\left(\frac{p-1}{p}\right)^2 \Lambda_2 \le \Lambda_p \le \Lambda_2. \tag{3.2}$$

Proof. The r.h.s. of this inequality is proved by contradiction: Assume that $\Lambda_p > \Lambda_2$ and substitute $\frac{\rho}{\rho_\infty} = |f|^{2/p}\left(\int_{\mathbb{R}^n} |f|^{2/p}\, d\rho_\infty\right)^{-1}$ (cf. (1.8)). A standard linearization argument (put $f^2 = 1 + \varepsilon w$ and take the limit $\varepsilon \to 0$) then implies a Poincaré inequality with the constant Λ_p which would contradict the sharpness of Λ_2 in (3.1).

For the l.h.s. of inequality (3.2) we estimate (using twice Jensen's inequality and then the Poincaré inequality):

$$\begin{aligned}
&\int_{\mathbb{R}^n} f^2\, d\rho_\infty - \left(\int_{\mathbb{R}^n} |f|^{\frac{2}{p}}\, d\rho_\infty\right)^{2(p-1)} \left(\int_{\mathbb{R}^n} f^2\, d\rho_\infty\right)^{\frac{2-p}{p}} \\
\le\ &\int_{\mathbb{R}^n} f^2\, d\rho_\infty - \left(\int_{\mathbb{R}^n} f\, d\rho_\infty\right)^{\frac{2}{p}2(p-1)} \left(\int_{\mathbb{R}^n} f\, d\rho_\infty\right)^{2\frac{2-p}{p}} \\
\le\ &\frac{1}{\Lambda_2} \int_{\mathbb{R}^n} D|\nabla f|^2\, d\rho_\infty .
\end{aligned}$$

This reformulation of (1.6) (just like in (1.11)) shows that $4\left(\frac{p-1}{p}\right)^2 \Lambda_2 \le \Lambda_p$. □

Next we shall show that the validity of a logarithmic Sobolev inequality implies the convex Sobolev inequalies (1.9) and (1.11). Part (*i*) of the following corollary is mainly due to Latała and Oleszkiewicz (Corollary 1 of [11]), with an improved constant for $\frac{3}{2} < p < 2$.

Corollary 5 *Let $D(x) > 0$ and let μ be a probability measure on $\mathbb{R}^d$ that gives rise to a logarithmic Sobolev inequality:*

$$\int f^2 \log\left(\frac{f^2}{\|f\|^2_{L^2(d\mu)}}\right) d\mu \leq \frac{2}{\Lambda_1} \int D|\nabla f|^2 \, d\mu \quad \forall\, f \in L^2(d\mu)\,. \tag{3.3}$$

Then:

(i) a convex Sobolev inequality *holds for any $p \in (1, 2]$:*

$$\int f^2 \, d\mu - \left(\int |f|^{2/p} \, d\mu\right)^p \leq \frac{\min\{2(p-1), 1\}}{\Lambda_1} \int D|\nabla f|^2 \, d\mu \quad \forall\, f \in L^2(d\mu)\,.$$

(ii) a refined convex Sobolev inequality *holds for any $p \in (1, 2]$:*

$$\int f^2 d\mu - \left(\int |f|^{\frac{2}{p}} d\mu\right)^{2(p-1)} \left(\int f^2 d\mu\right)^{\frac{2-p}{p}} \leq \frac{1}{\Lambda_1} \int D|\nabla f|^2 d\mu. \tag{3.4}$$

Proof. The function $p \mapsto \alpha(p) := p \log\left(\int |f|^{2/p} \, d\mu\right)$ is convex:

$$\alpha''(p) = \frac{4}{p^3} \, \frac{\left(\int |f|^{2/p} (\log|f|)^2 \, d\mu\right)\left(\int |f|^{2/p} \, d\mu\right) - \left(\int |f|^{2/p} \log|f| \, d\mu\right)^2}{\left(\int |f|^{2/p} \, d\mu\right)^2} \geq 0\,.$$

Thus $p \mapsto e^{\alpha(p)}$ is also convex and

$$p \mapsto \varphi(p) := \frac{e^{\alpha(1)} - e^{\alpha(p)}}{p-1}$$

is nonincreasing:

$$\varphi(p) \leq \lim_{q \to 1} \varphi(q) = \int f^2 \log\left(\frac{f^2}{\|f\|^2_{L^2(d\mu)}}\right) d\mu\,.$$

This proves that

$$\int f^2 \, d\mu - \left(\int |f|^{2/p} \, d\mu\right)^p \leq \frac{2(p-1)}{\Lambda_1} \int D|\nabla f|^2 \, d\mu\,.$$

On the other hand, using the linearization from the proof of Theorem 4 for (3.3) and using Hölder's inequality, $\left(\int f \, d\mu\right)^2 \leq \left(\int |f|^{2/p} \, d\mu\right)^p$, we also get

$$\int f^2 \, d\mu - \left(\int |f|^{2/p} \, d\mu\right)^p \leq \int f^2 \, d\mu - \left(\int f \, d\mu\right)^2 \leq \frac{1}{\Lambda_1} \int D|\nabla f|^2 \, d\mu\,.$$

Similarly, since the logarithmic Sobolev inequality (3.3) implies a classical Poincaré inequality, (ii) follows directly from Theorem 4. □

3.2 Holley-Stroock type perturbations

In Section 1 we presented the refined convex Sobolev inequality (1.6) for steady state measures $\rho_\infty = e^{-A(x)}$, whose potential $A(x)$ satisfies the *Bakry–Emery condition* (A2). We shall now extend that inequality for potentials $\widetilde{A}(x)$ that are bounded perturbations of such a potential $A(x)$. Our result generalizes the perturbation lemma of Holley and Stroock (*cf.* [10] for the logarithmic entropy ψ_1 and §3.3 of [2] for general admissible entropies).

For our subsequent calculations it is convenient to rewrite (1.6) as

$$k\left(\int_{\mathbb{R}^n} \psi\left(\frac{f^2}{\|f\|^2_{L^2(d\rho_\infty)}}\right) d\rho_\infty\right) \le \frac{2}{\lambda_1}\int_{\mathbb{R}^n} \frac{f^2}{\|f\|^4_{L^2(d\rho_\infty)}}\,\psi''\left(\frac{f^2}{\|f\|^2_{L^2(d\rho_\infty)}}\right) D|\nabla f|^2\, d\rho_\infty\,, \tag{3.5}$$

where we substituted

$$\frac{\rho}{\rho_\infty} = \frac{f^2}{\int_{\mathbb{R}^n} f^2\, d\rho_\infty}.$$

Theorem 6 *Assume that $\psi = \psi_p$ with some $1 < p < 2$ is a fixed entropy generator. Let $\rho_\infty(x) = e^{-A(x)}$, $\widetilde{\rho_\infty}(x) = e^{-\widetilde{A}(x)} \in L^1_+(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}\rho_\infty dx = \int_{\mathbb{R}^n}\widetilde{\rho_\infty} dx = M$ and*

$$\begin{aligned} \widetilde{A}(x) &= A(x) + v(x)\,, \\ 0 < a &\le e^{-v(x)} \le b < \infty\,, \qquad x \in \mathbb{R}^n. \end{aligned} \tag{3.6}$$

Let the given diffusion $D(x)$ be such that the convex Sobolev inequality (3.5) holds for all $f \in L^2(d\rho_\infty)$. Then a convex Sobolev inequality also holds for the perturbed measure $\widetilde{\rho_\infty}$:

$$\frac{1}{a^{p-1}}\,k\left(\frac{a^p}{b}\int_{\mathbb{R}^n} \psi\left(\frac{f^2}{\|f\|^2_{L^2}}\right) d\widetilde{\rho_\infty}\right) \le \frac{2}{\lambda_1}\int_{\mathbb{R}^n} \frac{f^2}{\|f\|^4_{L^2}}\,\psi''\left(\frac{f^2}{\|f\|^2_{L^2}}\right) D|\nabla f|^2\, d\widetilde{\rho_\infty} \tag{3.7}$$

for all nontrivial $f \in L^2(d\widetilde{\rho_\infty}) = L^2(d\rho_\infty)$. Here $\|f\|^2_{L^2}$ stands for $\|f\|^2_{L^2(d\widetilde{\rho_\infty})}$.

Note that the normalization of ρ_∞ and $\widetilde{\rho_\infty}$ implies $a \le 1$ and $b \ge 1$.

Proof. First we introduce the notations

$$\chi(x) := \frac{f^2(x)}{\|f\|^2_{L^2(d\rho_\infty)}}\,, \quad \tilde{\chi}(x) := \frac{f^2(x)}{\|f\|^2_{L^2(d\widetilde{\rho_\infty})}}\,, \quad \gamma := \frac{\chi}{\tilde{\chi}} = \frac{\|f\|^2_{L^2(d\widetilde{\rho_\infty})}}{\|f\|^2_{L^2(d\rho_\infty)}}\,,$$

and because of (3.6) we have $a \le \gamma \le b$.

We adapt the idea of [10, 2] and define for a fixed $f \in L^2(d\rho_\infty)$ the function

$$g(s) := s^p \int_{\mathbb{R}^n} \psi\left(\frac{f^2}{s}\right) d\widetilde{\rho_\infty}\,.$$

Since g attains its minimum at $s = \|f\|^2_{L^2(d\widetilde{\rho_\infty})}$, by differentiating w.r.t. s, we have

$$\begin{aligned} \|f\|^{2p}_{L^2(d\widetilde{\rho_\infty})} \int_{\mathbb{R}^n} \psi(\tilde{\chi})\, d\widetilde{\rho_\infty} = g(\|f\|^2_{L^2(d\widetilde{\rho_\infty})}) &\le g(\|f\|^2_{L^2(d\rho_\infty)}) \\ &\le b\, \|f\|^{2p}_{L^2(d\rho_\infty)} \int_{\mathbb{R}^n} \psi(\chi)\, d\rho_\infty\,, \end{aligned}$$

where we used the estimate (3.6).

Using the monotonicity of k and Assumption (3.5), this yields:

$$\begin{aligned} k\left(\frac{\gamma^p}{b}\int_{\mathbb{R}^n}\psi(\tilde{\chi})\,d\widetilde{\rho_\infty}\right) &\leq k\left(\int_{\mathbb{R}^n}\psi(\chi)\,d\rho_\infty\right) \\ &\leq \frac{2}{\lambda_1}\int_{\mathbb{R}^n}\frac{f^2}{\|f\|^4_{L^2(d\rho_\infty)}}\psi''(\chi)D|\nabla f|^2\,d\rho_\infty \\ &\leq \frac{2}{\lambda_1}\,\frac{\gamma^p}{a}\int_{\mathbb{R}^n}\frac{f^2}{\|f\|^4_{L^2(d\widetilde{\rho_\infty})}}\psi''(\tilde{\chi})D|\nabla f|^2\,d\widetilde{\rho_\infty}\ , \end{aligned} \tag{3.8}$$

where we again used (3.6) in the last estimate.

Since $\gamma/a \geq 1$, the convexity of k and $k(0) = 0$ imply:

$$\frac{\gamma^p}{a^p}\,k\left(\frac{a^p}{b}\int_{\mathbb{R}^n}\psi(\tilde{\chi})\,d\widetilde{\rho_\infty}\right) \leq k\left(\frac{\gamma^p}{b}\int_{\mathbb{R}^n}\psi(\tilde{\chi})\,d\widetilde{\rho_\infty}\right) .$$

Together with (3.8), this finishes the proof. □

Note that a *Holley–Stroock perturbation* of the usual convex Sobolev inequality (1.3) would lead – under the assumptions of Theorem 6 – to the inequality

$$\frac{a}{b}\,2\lambda_1\,e \leq |e'| \tag{3.9}$$

(*cf.* [2]). Since

$$\frac{a}{b}\,e < \frac{1}{a^{p-1}}\,k\left(\frac{a^p}{b}\,e\right) \qquad \forall\ \frac{a^p}{b}\,e > 0\ ,$$

Inequality (3.7) certainly improves (3.9).

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